Test-case number 3: Propagation of pure capillary standing waves (PA)

Pierre Lubin, TREFLE - UMR CNRS 8508, ENSCPB
Université Bordeaux 1, 33607 Pessac cedex, France
Phone: +33 (0)5 40 00 33 07, Fax : +33 (0)5 40 00 66 68
E-Mail: *lubin@enscpb.fr*

Edouard Canot, IRISA, Campus de Beaulieu, 35042 Rennes Cedex, France
Phone: +33 (0)2 99 84 74 89, Fax : +33 (0)2 99 84 71 71, E-Mail: *edouard.canot@irisa.fr*

1 Practical significance and interest of the test-case

Analytical solutions are provided here, developed for standing small-amplitude water waves. It provides a basis for applications to a series of numerical experiments. The interest consists here in predicting accurately the evolution of the interface of capillary waves in order to evaluate the coupling between inertial and viscous effects and estimating the effect of the numerical viscosity.

When simulating two-phase flows, it is important to evaluate the general accuracy and the validity of the numerical methods and numerical schemes used and the conservation laws of mass and energy in the computing domain. In particular, it is important to check that the behavior of the interface between two media is well taken into account, considering surface tension and viscous effects. As a matter of fact, capillary waves are similar to gravity waves but, firstly, they involve smaller scales, both in length and time. Secondly, They require a more difficult computation, because surface tension forces are based on the interface curvature, which needs to be accurately described. Thus, the results provided for pure capillary waves are considered, as initial conditions to simulate their propagations in constant depths over horizontal beds. The precision of the simulation is checked by comparing the free-surface shapes to theoretical values, including the predicted decay rate due to viscous effects.

2 Definitions and model description

The important parameters to describe waves are their length and height, and the water depth d over which they are propagating. The length of the wave, L, is the horizontal distance between two successive wave crests or two successive wave troughs. H is the height between the trough and the crest of the wave. The wave period, T, is the time required for two successive crests or troughs to pass a particular point. The speed of the wave, called the celerity c, is then defined as $c = L/T$. The water surface elevation η is the distance between the water surface and the mean water depth h.

Let us consider a standing small-amplitude wave with water surface displacement given by:

$$\eta(x,t) = \frac{H}{2}\cos(kx)\cos(\omega t), \tag{1}$$

with $\omega = 2\pi/T$ being the angular frequency of the wave, calculated from the dispersion relationship, $\omega^2 = gk\ \tanh(kd)$, and $k = 2\pi/L$ being the wave number. At $t = 0$, the water wave has a cosine shape, as shown in figure 1.

This is known as the linear wave theory, developed under the following assumptions. The fluid is supposed to be homogeneous and incompressible (density is constant), ideal or inviscid (lacks viscosity), the wave form is invariant in time and space (except its amplitude), the waves are two-dimensional and the sea bed is an horizontal, fixed, impermeable boundary which implies that the vertical velocity at the sea bed is zero. The restriction to small-amplitude implies that the ratio of the maximum elevation to the wavelength $H/L << 1$.

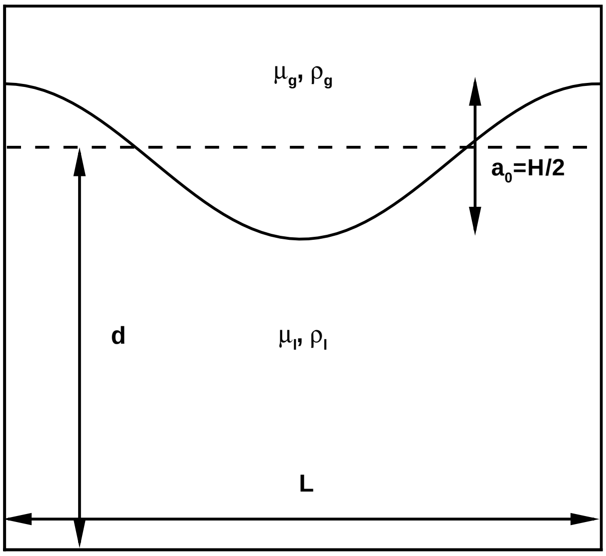

Figure 1: Initial conditions for the free-surface shape $\eta(x, 0)$.

Depending on the relative depth kd over which the waves are propagating, the water particle displacements are different. In *shallow water* ($d/L < 1/20$ or $kd < \pi/10$) and *intermediate depth* ($1/20 < d/L < 1/2$ or $\pi/10 < kd < \pi/2$), the water particle trajectory is an ellipse. In *deep water* ($d/L > 1/2$ or $kd > \pi/2$), the trajectories are circles decaying exponentially with depth.

According to the theoretical prediction for small-amplitude capillary waves (Lamb, 1932)[sec. 266], a generalized analytical value of the frequency ω_{th}, for finite depth, is given by:

$$\omega_{th}^2 = \frac{\sigma k^3}{\rho_l + \rho_g} \tanh(kd), \tag{2}$$

with σ being the constant surface tension, ρ_l and ρ_g being the densities.

Moreover, in the case where $\nu = \nu_l = \nu_g$ is the kinematic viscosity of both fluids, an analytical solution has developed by Prosperetti (1981) to calculate the evolution of the amplitude of a capillary wave. This solution takes into account the effects of the viscosity and the initial condition. In addition to the analytical value of ω_{th} (2), a dimensionless viscosity ϵ is defined:

$$\epsilon = \frac{\nu k^2}{\omega_{th}} \tag{3}$$

Prosperetti (1981) gives the solution for the shape of the interface:

$$\frac{\eta(x,t)}{\eta(x,0)} = a(t), \tag{4}$$

with $a(t)$ being the amplitude of the considered capillary wave. This amplitude is expressed as:

$$\begin{aligned}\frac{a(\tau)}{a_0} &= \frac{4(1-4\beta)\epsilon^2}{8(1-4\beta)\epsilon^2+1}\mathrm{erfc}(\sqrt{\epsilon\tau}) \\ &+\sum_{i=1}^{4}\frac{z_i}{Z_i}\frac{\omega_{th}}{{z_i}^2-\epsilon\omega_{th}}\exp\Big(({z_i}^2-\epsilon\omega_{th})\frac{\tau}{\omega_{th}}\Big)\mathrm{erfc}\Big(z_i\sqrt{\frac{\tau}{\omega_{th}}}\Big),\end{aligned} \tag{5}$$

with $\tau = \omega_{th}t$, and erfc being the complementary error function. z_i are solutions of the following equation:

$$\begin{aligned}&z^4 - 4\beta(\epsilon\omega_{th})^{\frac{1}{2}}(z^3 + 2(1-6\beta)(\epsilon\omega_{th})z^2 \\ &+4(1-3\beta)(\epsilon\omega_{th})^{\frac{3}{2}}z + (1-4\beta)(\epsilon\omega_{th})^2 + {\omega_{th}}^2 = 0,\end{aligned} \tag{6}$$

The coefficient Z_1 is given by $Z_1 = (z_2 - z_1)(z_3 - z_1)(z_4 - z_1)$, and the other coefficients Z_2, Z_3 and Z_4 are obtained by circular permutation of the subscripts. The parameter β is defined as:

$$\beta = \frac{\rho_l\rho_g}{(\rho_l+\rho_g)^2}, \tag{7}$$

In the case where ν_l and ν_g are being chosen with different values, the analytical solution (5) is no longer valid.

3 A series of test-cases

It is proposed to evaluate the numerical diffusion by simulating pure capillary waves ($g = 0$) propagating on the interface between two viscous fluids in a two-dimensional domain of length equal to the wavelength L, and to compare the numerical results with the analytical solutions developed previously.

The proposed numerical configuration is to consider an initial wave computed from the theory detailed before. The crest is located on both sides of the numerical domain ($x = 0$ and $x = L$, as shown in figure 1, symmetry boundary conditions being imposed on the lateral boundaries. Thus, at the instant $t = 0$, for $0 < x < L$, we have (Lamb, 1932)[sec. 250]:

$$\eta(x,0) = a_0\ \cos(kx), \tag{8}$$

with a_0 being the amplitude of the wave $a_0 = \frac{H}{2}$. There is no initial velocity, the fluid being at rest.

Non-viscous case

In the case of two fluids which viscosities are negligible ($\nu = \nu_l = \nu_g = 0$), capillary waves should not be damped and should oscillate with a constant frequency (2). It is so proposed to evaluate the variation of the ratio ω_{num} over ω_{th} as a function of the mesh size. The computation should then converge to this value of the frequency. However, the limit values obtained numerically will not be exactly equal to the theoretical ones: the effect of a numerical diffusion will then be highlighted.

As we are in the case where $\nu = \nu_l = \nu_g$, the amplitude of the oscillations $a(t)$ should also be plotted as a function of time and should be compared with the analytical solution given in (5). Duquennoy (2000) proposed the following parameters:

- $d/L = 0.5$, $H/L = 1$, $a/L = 2.7\ .10^{-2}$;
- $\rho_l/\rho_g = 1$, $\nu_l/\nu_g = 1$, with $k\nu_g/\omega_{th}a(t) << 1$ and $k\nu_l/\omega_{th}a(t) << 1$;
- $\sigma = 3.704\ .10^{-7}\ N.m^{-1}$.

Tests should be done with $k = \pi/L$, $2\pi/L$ and $4\pi/L$, with different mesh size. The accuracy of the results can be estimated by calculating the mean and strandard deviation between the numerical and analytical results.

Viscous case

In the contrary, when the viscosities are non-negligible and defined by μ_l, μ_g, the evolution of the interface at position $x = 0$ should be displayed and compared with the expected decay rate γ due to viscous effect (Lamb, 1932)[sec. 348]:

$$\gamma = 2\nu_l k^2 \tag{9}$$

This has been developed to estimate the effect of viscosity on free oscillatory waves on deep water by evaluating the energy dissipation, so it can provide a good approximation of damping since simulated waves are propagating in depths more than half the wavelength. The time evolution of the interface at position $x = 0$ should be plotted, as a function of time, compared to the predicted viscous envelope, $\eta(0,0)\exp(-\gamma t)$. The decay in time of the capillary wave should then be reproduced by the numerical simulation.

If we are in the case where $\nu = \nu_l = \nu_g$, the evolution of the amplitude $a(t)$ of the oscillations, given analytically in (5), should be compared to the previous curves plotted as a function of time.

Computations can be done with $\omega_{th} = 6.778$, $\epsilon = 6.472\ .10^{-2}$, $\rho_l/\rho_g = 1$, $\nu_l/\nu_g = 1$, and $k = 4\pi/L$ (Popinet & Zaleski, 1999). The accuracy of the results can be estimated by calculating the mean and standard deviation between the numerical and analytical results, as a function of the mesh size.

Summary of the required calculations for propagations of capillary waves

Simulations are to be performed with different values of the wave number $k = \pi/L$, $2\pi/L$ and $4\pi/L$. It is also required to check the conservation of mass during the simulations.

- In the case where $\nu = \nu_l = \nu_g$ and
  - ν_l and ν_g are negligible first;
  - ν_l and ν_g are then not negligible,

  the amplitude of the oscillations $a(t)$ should be plotted as a function of time and should be compared with the analytical solution (5) and the predicted viscous envelope $\eta(0,0)\exp(-\gamma t)$ (9). The ratio $\frac{\omega_{num}}{\omega_{th}}$ should be checked as a function of the mesh size.

- In the case where $\nu_l \neq \nu_g$ and
  - ν_l and ν_g are negligible first;
  - ν_l and ν_g are then not negligible,

  the amplitude of the oscillations $a(t)$ should be plotted as a function of time and should be compared with $\eta(0,0)\exp(-\gamma t)$, γ being evaluated from (9). The ratio $\frac{\omega_{num}}{\omega_{th}}$ should be checked as a function of the mesh size.

References

Duquennoy, C. 2000. *Développement d'une approche de simulation numérique directe de l'ébullition en paroi.* Ph.D. thesis, Institut National Polytechnique de Toulouse.

Lamb, H. 1932. *Hydrodynamics.* 6th edn. Cambridge University Press.

Popinet, P., & Zaleski, S. 1999. A front tracking algorithm for the accurate representation of surface tension. *Int. J. Numer. Meth. Fluids*, **30**, 775–793.

Prosperetti, A. 1981. Motion of two superposed viscous fluids. *Physics of Fluids*, **24**, 1217–1223.