Test-case number 4: Rayleigh-Taylor instability for isothermal, incompressible and non-viscous fluids (PA)

Édouard Canot, IRISA, Campus de Beaulieu, 35042 Rennes cedex, France
Phone: +33 (0)2 99 84 74 89, Fax: +33 (0)2 99 84 71 71, E-Mail: *edouard.canot@irisa.fr*

Stéphane Vincent, TREFLE - UMR CNRS 8508, ENSCPB
Université Bordeaux 1, 33607 Pessac cedex, France
Phone: +33 (0)5 40 00 27 07, Fax: +33 (0)5 40 00 66 68, E-Mail: *vincent@enscpb.fr*

1 Practical significance and interest of the test-case

The main interest of the present test-case is that it is very simple and should be treated by all numerical codes. It concerns a physical situation where gravity forces destabilize a flat interface, while, optionally surface tension forces tend to stabilize it. Numerical comparisons may be carried either in 2D or 3D geometry and concern essentially the precision of the interface position. If surface tension is present, this test can be used to measure the correctness of capillary forces, which depend on curvature values.

If computations are made with a general viscous model, Reynolds number must be very large (*e.g.* $Re \gtrsim 500$). Chandrasekhar (1961) and Yih (1965) also treated the case where viscous forces are present; viscosity doesn't change the stability criterion (3), but changes the amplification rate (2). This latter influence will not be discussed here but it should be generally small.

2 Definitions and physical model description

A closed box with vertical boundaries containing two immiscible liquids is considered. The heaviest liquid is above the other one and the interface between them is nearly flat, with gravity acting perpendicularly to this interface. We are interested in the time evolution of the interface, whose initial shape matches the most unstable mode.

Flow is supposed to be isothermal and incompressible. Moreover, the fact that viscous effects are neglected leads to a potential flow (*i.e.* irrotational). On vertical boundaries, the contact angle is constant, equal to $\frac{\pi}{2}$ (perfect waves reflection, or symmetry of the problem about the wall).

The physical properties are the following : (index u = upper, l = lower)

ρ_u, ρ_l : liquid densities ($\rho_u > \rho_l > 0$)

g : gravity ($g > 0$)

σ : surface tension between the two liquids ($\sigma \geq 0$)

By choosing the following scales, we turn the problem into a nondimensional form :

- length scale : L (half length of the box in the 2D case, or radius of the box in the 3D case)
- pressure scale : $(\rho_u - \rho_l)gL$

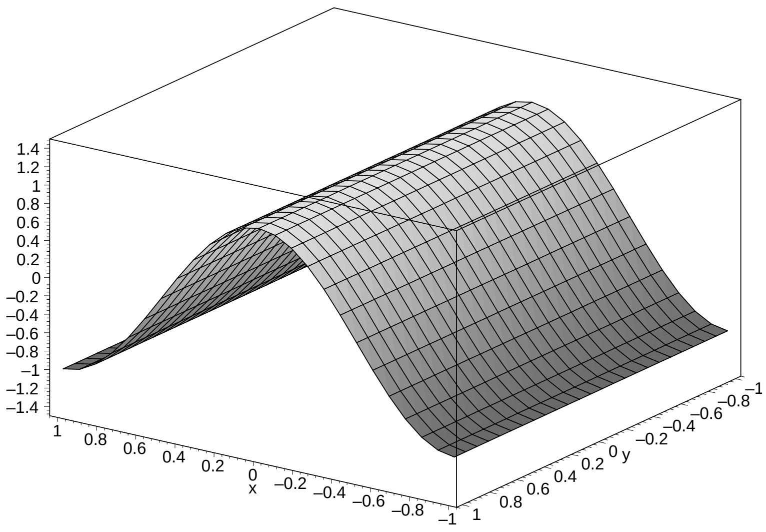

Figure 1: (4.1) — 2D plane case.

- velocity scale : $U = \sqrt{2A\,gL}$, where A is the Atwood number defined below (note that this choice has been found by requiring the forcing term of the problem, *i.e.* the gravity, to be of the order of inertia : $(\rho_u - \rho_l)gL = \frac{1}{2}(\rho_u + \rho_l)U^2$)

The non dimensional parameters are then :

- Atwood number : $A = \dfrac{\rho_u - \rho_l}{\rho_u + \rho_l}$ (which appears only in the time scale)
- Eötvös number : $Eo = \dfrac{(\rho_u - \rho_l)gL^2}{\sigma}$ (which is sometimes also called Bond number)

Hereafter, all variables are under the dimensionless form.

3 Test-case description

The following cases are studied :

(4.1) 2D plane case – rectangular box : $-1 < x < 1$, $-\frac{H}{L} < z < \frac{H}{L}$
initial shape of the interface (see figure 1) :

$$z = \varepsilon \cos(k\,x), \qquad k = \pi$$

(4.2) 3D axisymmetric case – circular cylinder box : $0 < r < 1$, $-\frac{H}{L} < z < \frac{H}{L}$
initial shape of the interface (see figure 2) :

$$z = \varepsilon J_0(k\,r), \qquad k = 3.83$$

(4.3) 3D case – same box as in case (4.2)
initial shape of the interface (see figure 3) :

$$z = \varepsilon \cos(\theta) J_1(k\,r), \qquad k = 1.84$$

N.B. The above numerical values, i.e. 3.83 and 1.84, are the first maximum (out of the axis) of the Bessel functions J_0 and J_1, respectively.

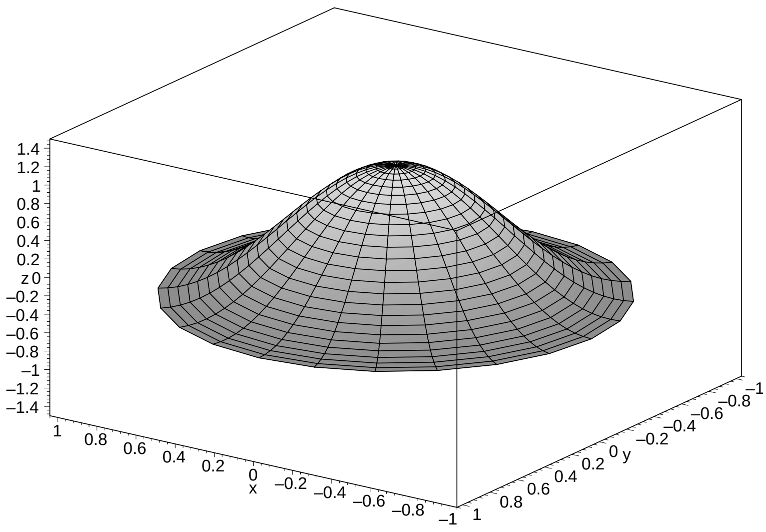

Figure 2: (4.2) — 3D axisymmetric case.

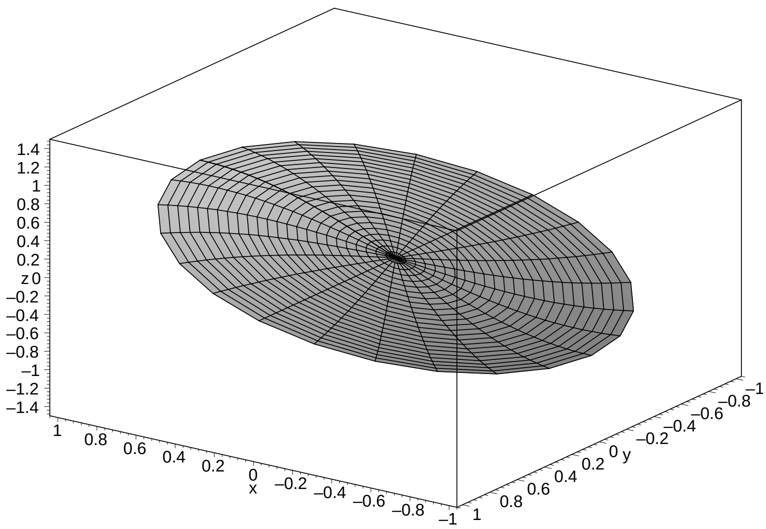

Figure 3: (4.3) — 3D case.

3.1 Analytical solutions

The interface shapes used in this study come from the solution of linearized equations (the $\rho\mathbf{u}^2$ term in Bernoulli equation is neglected) for a quasi-horizontal surface of infinite extend :

$$\frac{H}{L} \gg 1, \qquad \varepsilon \ll 1$$

As long as the flow remains linear, in each of the three studied cases, the interface keeps the same shape during its deformation and its amplitude reads :

$$\varepsilon\, e^{\alpha\, t} \tag{1}$$

The linear amplification rate α is defined as :

$$\alpha^2 = \frac{k}{2}\left(1 - \frac{k^2}{Eo}\right) \tag{2}$$

where k is the dimensionless wave number defined above in each of the three sub-cases.

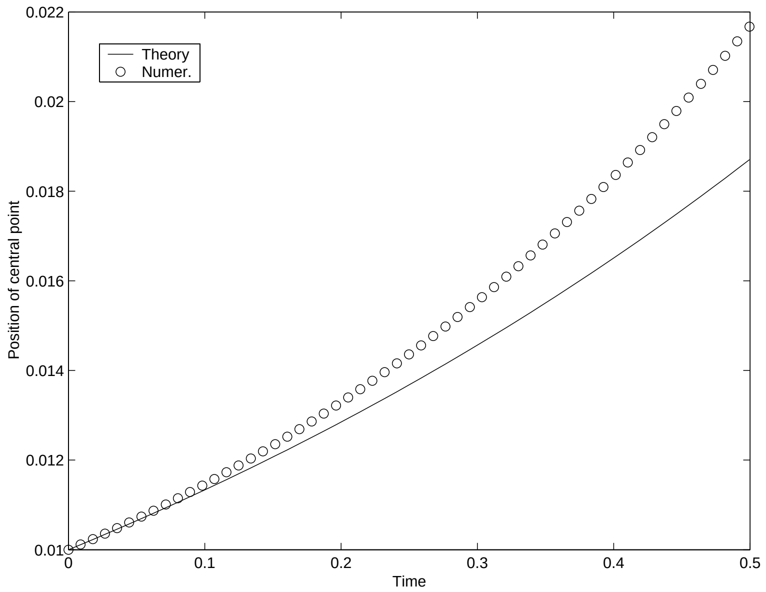

Figure 4: BEM simulation of the Rayleigh-Taylor instability. Position of the central point.

Instability occurs only when the numerical value of α^2 is strictly positive, *i.e.* when :

$$Eo > k^2 \tag{3}$$

(note: if $\alpha^2 < 0$, periodic solutions are obtained; it can be another test-case!)

3.2 Test comparison criteria

If the flow remains in the linear domain, comparisons with numerical codes may be focused: firstly, on the interface shape, secondly, on the amplification rate (α coefficient). The amplitude must be small enough during the whole computation; the appendix provides some guidelines for such a practical condition. Moreover, in the linear regime, space and time convergence studies can be accurately leaded with the present test case. For example, the difference between analytical and numerical amplification rates ε can be tested.

Other numerical simulations can be found in Tryggvason (1988) and Kerr (1988).

Appendix: an example of a numerical test for the 2D case

To compute the Rayleigh-Taylor instability in the 2D case, we make use of the BEM method (details in Machane & Canot, 1997) which is well adapted to this case. Following values are chosen :

- $\dfrac{H}{L} = 3.0$
- $\varepsilon = 0.01$
- $A = 0$ (one liquid above vacuum)
- $Eo = +\infty$ (no surface tension), and so $\alpha^2 = \dfrac{\pi}{2}$

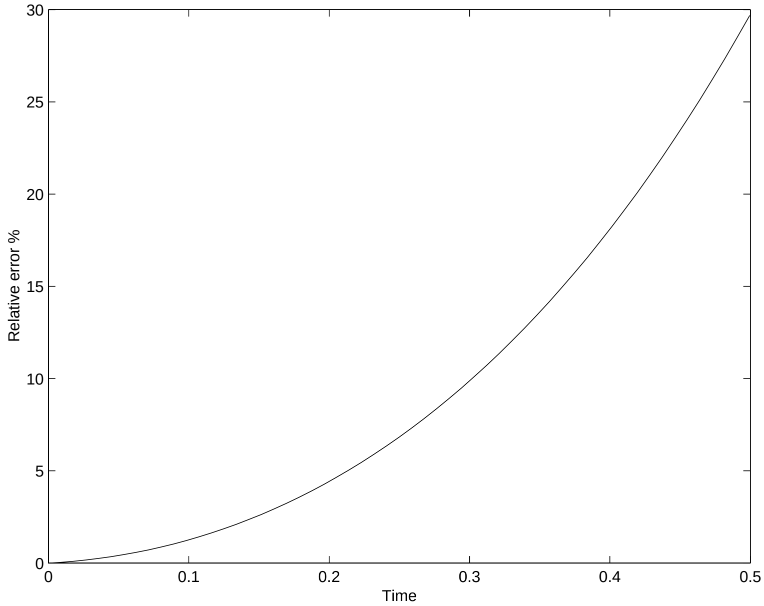

Figure 5: BEM simulation of the Rayleigh-Taylor instability. Discrepancy between computed results and the linear theory shown in figure 4.

The theoretical solution is :

$$z_{theo}(x,t) = \varepsilon \cos(\pi x) e^{\alpha t}$$

and, consequently, initial conditions are set as follows :

$$z_{theo}(x,0) = \varepsilon \cos(\pi x)$$

$$\varphi_{theo}(x,0) = \frac{\alpha \varepsilon}{\pi} \cos(\pi x)$$

where φ_{theo} is the velocity potential on the free surface. 160 nodes have been equally spaced along the boundary.

Figures 4 and 5 show that the discrepancy between linear theory, $z_{theo} + (x = 0, t)$, and numerical results, $z_{num} + (x = 0, t)$, quickly increases, so that the simulation should be limited to short duration. Especially, in figure 5, the relative error is computed as:

$$\text{relative error} = \frac{z_{num}(x = 0, t) - z_{theo}(x = 0, t)}{\varepsilon}$$

References

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