Test-case number 5: Oscillation of an inclusion immersed in a quiescent fluid (PA)

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1 Practical significance and interest of the test-case

The physical situation of interest considers the oscillations of a fluid inclusion, *i. e.* a bubble or a droplet, imbedded in another fluid. The analytical results to be used as a reference are relative to inviscid fluids, therefore, this test-case is intended to provide a verification of the correct balance between surface tension and inertial effects which are the only effects which control the fluid motion. In addition, since the analytical results are non-dissipative, this test-case provides a tool to estimate the rate of the energy dissipation due to the modeling method.

2 Definitions and model description

In this two-fluid problem, the following notations are introduced to describe the fluids and interface physical and transport properties. The two fluids are considered as non-miscible and the motion of the inclusion is assumed to be axisymmetric. For the sake of convenience the z axis is considered as the vertical axis though gravity is not included in the model. The radial coordinate r is considered as horizontal. The axisymmetric inclusion is immersed in a infinite medium. The inclusion and the domain are defined by the following dimensions.

- The inclusion equivalent diameter is defined as the diameter of the sphere which volume is equal to that of the actual inclusion. Let D be this diameter and R the corresponding radius.
- The calculation domain has a height h along the z axis and a width L along the r axis. It may also be spherical with a radius a. The recommended dimensions of the domain will be discussed later.

The physical properties should correspond roughly to an inertia dominated flow with a very low viscosity for the gas and the liquid. This means that the capillary numbers based on the physical properties of each phase are very small. They are defined by,

$$Ca_k \triangleq \frac{\rho_k}{\mu_k}\sqrt{\frac{D\sigma}{\rho_k}}, \tag{1}$$

where the subscript $k = L, G$ denotes respectively the liquid or the gas phase, ρ_k is the fluid density, μ_k is the dynamical viscosity and σ is the surface tension between the gas and the liquid. Therefore, it is assumed that viscosity effects vanish when the following condition is satisfied:

$$Ca_k \gg 1. \tag{2}$$

The reader is free to use any physical property values provided that they satisfy (2).

The model to be used is the Navier-Stokes equations for both phases and the physical properties are assumed to remain uniform and constant. In particular, thermal effects are neglected and the energy equation is not considered. Potential flow should be the most relevant limiting case of this situation.

Rayleigh in 1879 derived the frequency of oscillation of such an inclusion. This derivation is based on the potential flow theory an is valid for any arbitrary inner and outer values of the density. This is shown for example by Lamb (1975, p. 475, equation 10). The result is obtained by expanding the inner and the outer potential of velocity on the complete set of eigenfunctions of the Laplace equation in spherical coordinates. The result is exact though its validity is limited since it relies on a linearized momentum jump equation at the interface where the dynamical effects, which are second order in the deviation from sphericity, have been neglected. The eigenfunctions are given by the following expression,

$$\phi_{nm} = \left\{ \begin{array}{c} r^n \\ \dfrac{1}{r^{n+1}} \end{array} \right\} \left\{ \begin{array}{c} Y^o_{mn}(\theta,\varphi) \\ \\ Y^e_{mn}(\theta,\varphi) \end{array} \right\} \text{ with } n \geqslant m \geqslant 0, \tag{3}$$

where Y^o_{mn}, and Y^e_{mn} are respectively the odd and even surface spherical harmonics of first kind as defined by Morse & Feshbach (1953, p. 1264, eq. 10.3.25) and θ is the angle with the z axis. These harmonics are related to the Legendre functions, P_n^m, by the following relations,

$$Y^o_{mn} = \sin(m\varphi) P_n^m(\cos\theta), \tag{4}$$

$$Y^e_{mn} = \cos(m\varphi) P_n^m(\cos\theta). \tag{5}$$

The spherical harmonics (4) and (5) are also the eigenmodes of deformation of the inclusion. The interesting result is that their frequency of oscillation does not depend on the azimuthal mode number m but only on n. This test-case can therefore be used for axisymmetric cases and really 3D flows as well. In particular, the axisymmetric modes are obtained for $m = 0$ where the Legendre functions becomes the Legendre polynomials denoted by P_n.

The angular frequency of the oscillations is obtained for each mode by using the linearized momentum jump at the interface; it is given by,

$$\omega_n^2 = \frac{n(n+1)(n-1)(n+2)}{(n+1)\rho_I + n\rho_O} \frac{\sigma}{R^3} \tag{6}$$

where ρ_I and ρ_O are respectively the inner and outer fluid density. It is reminded that the period, T_n, and the frequency, f_n, are related to the angular frequency, ω_n, by:

$$T_n = \frac{2\pi}{\omega_n}, \tag{7}$$

$$f_n = \frac{\omega_n}{2\pi}. \tag{8}$$

Some additional intermediate results are of interest to set the size of the calculation domain and appreciate the necessary smallness of the oscillations. Let us consider a harmonic oscillation of the inclusion along one of its eigenmodes,

$$r = R + \epsilon Y_{mn} \sin(\omega t + u), \tag{9}$$

where ϵ is the amplitude of the oscillation, u is an arbitrary phase and Y_{mn} is one of the spherical harmonics (4) or (5). The analysis shows that the corresponding potentials have the following form,

$$\phi_I = -\frac{\epsilon\omega R}{n} Y_{mn} \frac{r^n}{R^n} \cos(\omega t + u), \tag{10}$$

$$\phi_O = \frac{\epsilon\omega R}{n+1} Y_{mn} \frac{R^{n+1}}{r^{n+1}} \cos(\omega t + u). \tag{11}$$

The corresponding radial velocities are given by,

$$v_I = -\epsilon\omega R Y_{mn} \frac{r^{n-1}}{R^n} \cos(\omega t + u), \tag{12}$$

$$v_O = \epsilon\omega R Y_{mn} \frac{R^{n+1}}{r^{n+2}} \cos(\omega t + u), \tag{13}$$

whereas the corresponding pressure fields are given by,

$$p_I = \frac{\epsilon\rho_I\omega^2 R}{n} Y_{mn} \frac{r^n}{R^n} \sin(\omega t + u), \tag{14}$$

$$p_O = -\frac{\epsilon\rho_O\omega^2 R}{n+1} Y_{mn} \frac{R^{n+1}}{r^{n+1}} \sin(\omega t + u). \tag{15}$$

Since the fluids are assumed to be incompressible, there is no point in comparing a pressure fluctuation to an absolute value of the pressure. However, a calculation domain sizing argument could be that the pressure fluctuation at a distance a of the inclusion center is much smaller than its maximum value reached near the interface:

$$\frac{p_O(r = a)}{p_O(r = R)} = \frac{R^{n+1}}{a^{n+1}} \ll 1. \tag{16}$$

This ratio represents the order of magnitude of the error on the pressure induced by a calculation domain of size a at the boundary of which the pressure is imposed uniform. For the second order deformation, $n = 2$, which represents an ellipsoidal-spherical oscillation, the error decreases as the cube of the domain size. If the calculation domain is limited by the available computing capacity, it is recommended to select a mode compatible with the size of the domain according to (16).

On the other hand, if the domain is assumed impermeable the same reasoning with the velocity leads to the following condition:

$$\frac{v_O(r = a)}{v_O(r = R)} = \frac{R^{n+2}}{a^{n+2}} \ll 1. \tag{17}$$

Therefore, it seems more favorable to work with a zero velocity than a uniform pressure at the outer boundary.

The linearization of the problem introduces some limitations on the amplitude of the oscillations. The first one comes from the transfer of the boundary condition on the undistorted shape of the inclusion and the second from discarding the kinetic term in the Bernoulli equation used to express the momentum jump at the interface. Intuitively, the second source of error should be the most important. It requires that the corresponding terms in (12), (13) and (14), (15) are such that,

$$\frac{1}{2}\rho_k v_k^2 \ll p_k. \tag{18}$$

Using the following identity,

$$\int_{-1}^{1} P_n^l(z) P_n^m(z)\,\mathrm{d}z = \delta_{lm}(n+\frac{1}{2})^{-1}\frac{(n+m)!}{(n-m)!}, \tag{19}$$

where δ_{lm} is the Kronecker symbol, the following condition on the oscillation amplitude must be satisfied,

$$\frac{1}{2}\frac{\rho_k v_k^2}{p_k} \lesssim \frac{n+1}{4}\left(n+\frac{1}{2}\right)^{-\frac{1}{2}}\frac{\epsilon}{R}. \tag{20}$$

It is recommended to select a value of ϵ which corresponds to an error in the pressure of the same magnitude than that induced by the finite size of the calculation domain. For example, if one considers the dynamical pressure perturbation induced by the interface motion to be small with respect to the static pressure perturbation when it is 100 times smaller for the axisymmetric mode $n = 2$, then (20) shows that ϵ should be smaller than 1/30 of the initial radius R ($\frac{1}{2}\rho_k v_k^2/p_k < 1/100$ is achieved when $\epsilon/R < 1/30$ according to equation 20) .

3 Numerical settings, initial and boundary conditions

It is recommended to start the calculation with a distorted spherical shape by adding several perturbations of the kind of (9) with $m = 0$ for the axisymmetric case. It is further recommended to consider a similar amplitude for all the selected modes. In this latter case, the Legendre functions values at selected values of the parameter $z = \cos\theta$ are easy to calculate by using the following recurrence formulas (Abramovitch & Stegun, 1965, eq. 8.5.1)

$$P_n^{m+1}(z) = \frac{-1}{\sqrt{1-z^2}}\left((n-m)zP_n^m(z) - (n+m)P_{n-1}^m(z)\right), \tag{21}$$

$$(n-m+1)P_{n+1}^m(z) = (2n+1)zP_n^m(z) - (n+m)P_{n-1}^m(z), \tag{22}$$

where the following initial terms are given by,

$$P_0^0 = 1 \tag{23}$$

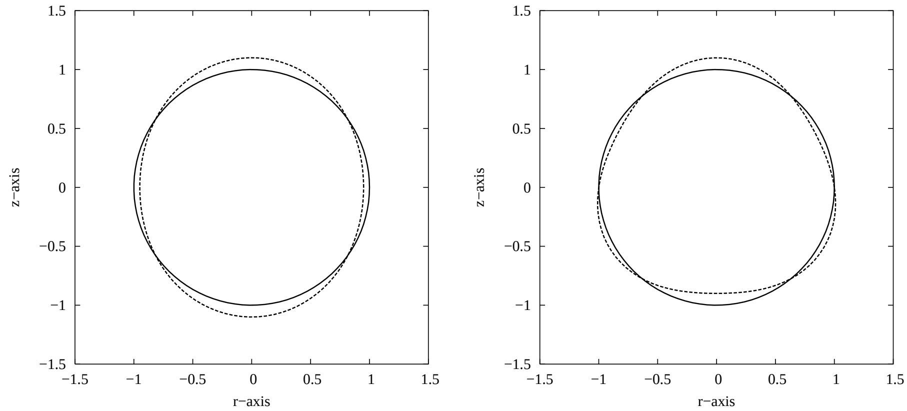
$$P_1^0 = \cos\theta, \quad P_1^1 = \sin\theta \tag{24}$$

$$P_2^0 = \frac{1}{4}(3\cos 2\theta + 1), \quad P_2^1 = \frac{3}{2}\sin 2\theta, \quad P_2^2 = \frac{3}{2}(1-\cos 2\theta) \tag{25}$$

The test-case does not specify the shape nor the dimensions of the outer boundary of the domain. The error estimates (16), (20) or (17), (20) must be used to justify the considered choices.

4 Requested calculations

It is proposed to calculate the time evolution of the shape of an initially non spherical inclusion. A possible postprocessing method could be to Fourier-analyze the position, $z(t)$, of the top intersection of the interface with the z axis and to consider the relative deviation of the eigenmodes frequency, $\Delta f_n/f_n$, from (6) and (8). Another suggestion consists in



(a) $n = 2$. (b) $n = 3$.

Figure 1: Plot of the two eigenmodes $n = 2$ and $n = 3$. The length unit is the radius, R, and the relative amplitude of each mode is set to $\epsilon/R = 0.1$. The equation of the surface is $r = 1 + 0.1P_n(\cos\theta)$ where $P_2 = 1/4(3\cos 2\theta + 1)$ and $P_3 = 1/8(5\cos 3\theta + 3\cos\theta)$.

determining the instantaneous amplitude of the eigenmodes which can be calculated by projecting the instantaneous shape, $\widetilde{r}(\theta, t)$, or $r(z, t)$ on the eigenmodes according to,

$$\eta_n(t) = \left(n + \frac{1}{2}\right) \int_0^{\pi} \widetilde{r}(\theta, t) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta \tag{26}$$

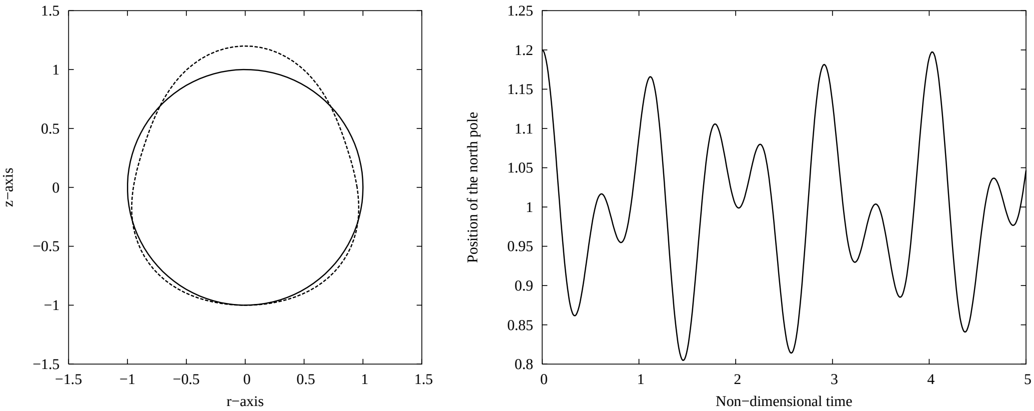
where the orthogonality of the Legendre functions (19) has been utilized. If the calculation is correct, the amplitude time trace, $\eta_n(t)$, should be a perfect cosine curve with an amplitude, ϵ_n, and phase, u, consistent with the mode selected (9) and a frequency equal to that given by (6) and (8). The time evolution of the eigenmodes amplitude must be provided to assess the diffusive nature of the method.

In addition to this results analysis the following is proposed. Considering that the volume of the inclusion should remain strictly constant, it is requested to compute its time evolution.

5 An illustrative example

This section provides an illustration of a possible initial shape to start with and shows the time trace of the north pole displacement which is expected. The first significant eigenmode for this test case is $n = 2$. Indeed, it is easy to show that the eigenmodes $n = 0, 1$ are, in the frame of our approximations, two equilibrium states. The $n = 0$ mode does not preserve the volume of the inclusion and it is therefore irrelevant; whereas the the mode $n = 1$ does it though the frequency relation (6) shows clearly that it is another equilibrium state. The figure 1 shows the two modes $n = 2$ and $n = 3$.

The figure 2(a) shows the superposition of the two modes of figure 1; whereas the figure 2(b) shows the expected evolution of the inclusion north pole position. The initial shape



(a) Superposition of modes 2 and 3. (b) Time trace of the north pole position.

Figure 2: Initial shape of the inclusion obtained by adding the two eigenmodes of figure 1. Time trace of the position $z(t)$ of the inclusion north pole. This configuration is relative to a bubble ($\rho_I = 0$). The length scale is the bubble radius R and the time scale is the period of oscillation of the mode $n = 2$.

is given in non-dimensional form by,

$$\frac{r}{R} = 1 + 0.1\left(\frac{1}{4}(3\cos 2\theta + 1) + \frac{1}{8}(5\cos 3\theta + 3\cos\theta)\right). \tag{27}$$

The position of the north pole of the inclusion is given by,

$$\frac{z(t)}{R} = 1 + 0.1(\cos 2\pi\tau + \cos 2\Omega\pi\tau), \tag{28}$$

where $\tau = t/T$ is a non dimensional time, and T the period of oscillation of the second eigenmode of an empty bubble ($\rho_I = 0$) surrounded by a liquid of density ρ_O. The period of oscillation of mode 2 is given by,

$$T = \frac{2\pi}{\omega_2},\ \omega_2^2 = 12\frac{\sigma}{\rho_O R^3}, \tag{29}$$

and Ω is the ratio of the two considered eigenmodes angular frequency obtained by,

$$\Omega = \frac{\omega_3}{\omega_2} = \sqrt{\frac{10}{3}}. \tag{30}$$

As shown by figure 2(b), the time evolution of the position of the north pole is not periodic as suggested by the non rational value of Ω given by (30).

6 Additional information for 2D calculations.

It is of interest to mention that analogous developments are available for purely 2D cases (Toutant, 2003). The analogous of equation (6) in two dimensions is the following,

$$\omega_n^2 = \frac{n(n+1)(n-1)}{(\rho_I + \rho_O)}\frac{\sigma}{R^3} \tag{31}$$

References

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