Test-case number 7b: Isothermal vaporization due to piston aspiration (PA)

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1 Practical significance and interest of the test-case

An analytical solution of a one-dimensional isothermal liquid-vapor phase-change problem is provided. In models dedicated to the description of multiphase flows, an important part is the modeling of the interfaces which separate the phases of the system. The equations of motion of the interfaces consist in balance equations (mass, momentum, energy and entropy) and closure relations. However, in general, these equations of motion are simplified by assuming some sort of equilibrium. For instance, it is generally assumed that the Laplace relation, which relates the pressure of the bulk phases to the curvature of the interface at equilibrium, is satisfied locally all along the interfaces. In this case, the general momentum balance equation at the interface is simplified by assuming local equilibrium at the interface.

If the system under consideration is a two-phase system made of liquid and vapor phases of a pure substance, phase-change might occur at an interface. This phase-change (evaporation or condensation) corresponds to a reaction of the system placed in a non-equilibrium state to recover an equilibrium state. For instance, let us consider a liquid-vapor system for which the pressure is assumed to be constant and uniform and let us heat up the vapor phase. Due to thermal conduction in the vapor phase, the temperature will increase, in particular at the interface. If local thermodynamic equilibrium is assumed at the interface, the temperature of the interface must remain constant and equal to the saturation temperature corresponding to the imposed pressure. To keep the interface at a constant temperature (*i.e.* at local thermodynamic equilibrium), evaporation occurs: the latent heat absorbed during evaporation prevents the interface temperature from rising up.

The local thermodynamic equilibrium condition is generally imposed in terms of the interface temperature as described above. However, the relation $T_i = T^{sat}(P_i)$[1], where T_i is the local interface temperature, P_i is the local interface pressure and T^{sat} is the corresponding saturation temperature, can be inverted: $P_i = P^{sat}(T_i)$, where P^{sat} is the saturation pressure. Even though these two conditions are equivalent, they correspond to very different situations: in the first case, it is assumed that the pressure is imposed and that phase-change is mainly driven by thermal effects, whereas in the second case, it is assumed that the temperature is imposed and that phase-change is mainly driven by dynamic or pressure effects.

Such a situation corresponds to the physical system sketched in figure 1: different velocities are imposed to the liquid and vapor phases, which creates a pressure gradient within the bulk phases, which tends to let the interface pressure be different from the saturation pressure, which then triggers phase-change.

The present test-case gives an analytical solution of a problem for which phase-change is driven by dynamic effects and not by thermal effects. This solution might be used to test the capability of a numerical method to simulate liquid-vapor phase-change due to dynamic effects and not thermal effects.

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[1] Curvature effects are neglected here for the sake of simplicity.

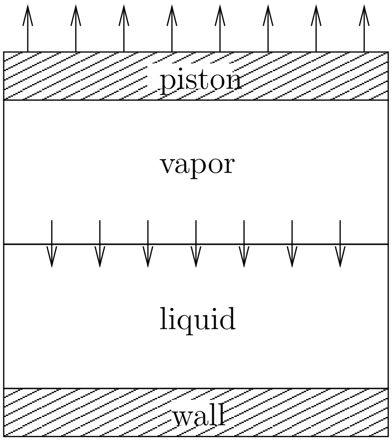

Figure 1: Sketch of a one-dimensional isothermal phase-change problem.

2 Definitions and model description

Let us consider a *liquid-vapor* system of a *pure substance*. The system is assumed to be *one-dimensional* and the interface to be *flat*; therefore only the direction normal to the interface, say z, is relevant. The system is also assumed to be *isothermal*. Physically, this means that the thermal conductivity is very large and that energy is supplied by thermal conduction where needed (mainly at the interface) instantly. No volumetric force is applied to the system. The equations of motion of the system are the following.

- Within the bulk phases

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial z} = 0 \tag{1}$$

$$\frac{\partial (\rho V)}{\partial t} + \frac{\partial (\rho V^2)}{\partial z} = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial z}\left(\frac{4}{3}\mu\frac{\partial V}{\partial z}\right) \tag{2}$$

  where ρ is the density, V is the velocity, P is the pressure and μ is the dynamic viscosity.

  Note that, if the bulk phases are assumed to be *incompressible*, which is a fair approximation as long as the velocity is small compared to the sound speed, the system (1)-(2) reads

$$\frac{\partial V}{\partial z} = 0 \tag{3}$$

$$\rho\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial z}\right) = -\frac{\partial P}{\partial z} \tag{4}$$

- At the interface (superscript i)

$$\rho_v^i\left(V_v^i - V^i\right) = \rho_l^i\left(V_l^i - V^i\right) \tag{5}$$

$$P_l^i = P^{sat} + \frac{\left(\dot{m}^i\right)^2}{2}\left(\frac{1}{\rho_v^i} - \frac{1}{\rho_l^i}\right) \tag{6}$$

$$P_v^i = P^{sat} - \frac{\left(\dot{m}^i\right)^2}{2}\left(\frac{1}{\rho_v^i} - \frac{1}{\rho_l^i}\right) \tag{7}$$

where V^i is the speed of displacement of the interface and $\dot{m}^i$ is the mass flux across the interface, which is defined by

$$\dot{m}^i = \rho_v^i\left(V_v^i - V^i\right) = \rho_l^i\left(V_l^i - V^i\right) \tag{8}$$

The subscripts v and l represent the vapor and liquid phases respectively.
Equations (6) and (7) correspond to the local thermodynamic condition at the interface (*e.g.* Ishii, 1975, p. 39 and Jamet *et al.* , 2001, p. 636). Note that the terms in $\left(\dot{m}^i\right)^2$ correspond to the effect of the kinetic energy.

- The boundary conditions are the following:

$$\rho V(z = 0, t) = \dot{m}_l \tag{9}$$

$$\rho V(z = L, t) = \dot{m}_v \tag{10}$$

$$\frac{\partial V}{\partial z}(z = 0, t) = 0 \tag{11}$$

$$\frac{\partial V}{\partial z}(z = L, t) = 0 \tag{12}$$

where L is the length of the system and $\dot{m}_l$ and $\dot{m}_v$ are two different *constants* which represent respectively the liquid mass flux entering the system and the vapor mass flux exiting the system. Figure 1 corresponds to the case $\dot{m}_l = 0$ and $\dot{m}_v > 0$.
Note that the boundary conditions (11) and (12) are necessary only if the bulk phases are compressible.

2.1 Incompressible model

As a first approximation, let us consider the *incompressible* model. In this case, the density is uniform within each bulk phase by definition of the incompressibility, and it is straightforward to show that the velocity and the pressure are also uniform within each bulk phase. The value of the bulk velocities depends on the imposed mass fluxes on the system boundaries (9) and (10). The value of the bulk pressures is determined by the interface local equilibrium conditions (6) and (7). For general boundary conditions $\dot{m}_l$ and $\dot{m}_v$, one gets:

$$V_l = \frac{\dot{m}_l}{\rho_l} \tag{13}$$

$$V_v = \frac{\dot{m}_v}{\rho_v} \tag{14}$$

$$P_l = P^{sat} + \frac{\rho_l\,\rho_v}{2\left(\rho_l - \rho_v\right)}\left(V_v - V_l\right)^2 \tag{15}$$

$$P_v = P^{sat} - \frac{\rho_l \, \rho_v}{2\,(\rho_l - \rho_v)} \, (V_v - V_l)^2 \tag{16}$$

The speed of displacement of the interface V^i and the interfacial mass flux $\dot{m}^i$ are given by

$$V^i = \frac{\dot{m}_l - \dot{m}_v}{\rho_l - \rho_v} \tag{17}$$

$$\dot{m}^i = \frac{\rho_l \, \dot{m}_v - \rho_v \, \dot{m}_l}{\rho_l - \rho_v} \tag{18}$$

Equation (17) shows that *the speed of displacement of the interface is constant*, which implies that, *in the frame of reference linked to the interface, the problem is stationary.*

Equation (18) shows that the interfacial mass flux $\dot{m}^i$ is constant and that its value depends only on the dynamic boundary conditions imposed through $\dot{m}_v$ and $\dot{m}_l$ and not on any thermal effect since the temperature is assumed to be constant and uniform.

It is worth noting that the value of the pressure within the bulk phases depend on the condition of local thermodynamic equilibrium at the interface as shown by equations (15) and (17).

2.2 Inviscid compressible model

We showed in the previous section that, when the bulk phases are incompressible, the interface moves at a constant speed and that the problem is therefore stationary in the framework linked to the interface. Let us therefore consider this framework and assume that the flow in the bulk phases is *stationary.* For the sake of simplicity, the bulk phases are assumed to be *inviscid.* The equations of motion (1)-(2) read

$$\frac{\partial(\rho V)}{\partial z} = 0$$

$$\frac{\partial(\rho V^2)}{\partial z} = -\frac{\partial P}{\partial z}$$

The constants which appear after integration are determined using the conditions (5), (6), (7), (9) and (10). One gets

$$(\rho V)_l \, (z) = \dot{m}_l \tag{19}$$

$$(\rho V)_v \, (z) = \dot{m}_v \tag{20}$$

$$\left(P + \rho V^2\right)_l (z) = P^{sat} + \frac{1}{2} \, \frac{\rho_l^i \, \rho_v^i}{\rho_l^i - \rho_v^i} \left(V_l^i - V_v^i\right)^2 + \rho_l^i \, \left(V_l^i\right)^2 \tag{21}$$

$$\left(P + \rho V^2\right)_v (z) = P^{sat} - \frac{1}{2} \, \frac{\rho_l^i \, \rho_v^i}{\rho_l^i - \rho_v^i} \left(V_l^i - V_v^i\right)^2 + \rho_v^i \, \left(V_v^i\right)^2 \tag{22}$$

In equations (19)-(22), the densities and the velocities of the liquid and vapor phases at the interface are unknown. Since the bulk phases are compressible, the densities ρ_l^i and ρ_v^i are solution of the following system of equations

$$P_l(\rho_l^i) + \rho_l^i \left(V_l^i\right)^2 = P^{sat} + \frac{1}{2}\,\frac{\rho_l^i\,\rho_v^i}{\rho_l^i - \rho_v^i}\left(V_l^i - V_v^i\right)^2 + \rho_l^i\,\left(V_l^i\right)^2 \tag{23}$$

$$P_v(\rho_v^i) + \rho_v^i \left(V_v^i\right)^2 = P^{sat} - \frac{1}{2}\,\frac{\rho_l^i\,\rho_v^i}{\rho_l^i - \rho_v^i}\left(V_l^i - V_v^i\right)^2 + \rho_v^i\,\left(V_v^i\right)^2 \tag{24}$$

in which the velocities V_l^i and V_v^i are expressed by

$$V_l^i = \frac{\dot{m}_l}{\rho_l^i} \tag{25}$$

$$V_v^i = \frac{\dot{m}_v}{\rho_v^i} \tag{26}$$

It is straightforward to show that the expressions for the speed of displacement of the interface and the interfacial mass flux are the following

$$V^i = \frac{\dot{m}_l - \dot{m}_v}{\rho_l^i - \rho_v^i} \tag{27}$$

$$\dot{m}^i = \frac{\rho_l^i\,\dot{m}_v - \rho_v^i\,\dot{m}_l}{\rho_l^i - \rho_v^i} \tag{28}$$

2.3 Physical relevance of an isothermal phase-change problem

The physical situation sketched in figure 1 corresponds to a situation in which the liquid-vapor phase-change is imposed through a dynamic effect and not through a thermal effect. However, the fact that phase-change can occur at a constant temperature is less intuitive. Indeed, interfacial mass transfer is possible only if sufficient energy is provided at the interface, since the interfacial energy balance equation reads

$$q_l^i - q_v^i = \dot{m}^i\,\mathcal{L}$$

where q is the conductive heat flux and $\mathcal{L}$ is the latent heat of vaporization (in which kinetic energy effects are taken into account).

Given this interfacial energy balance equation, the present model is such that any energy needed at the interface for mass transfer is provided instantly. This corresponds to infinite values of the bulk thermal conductivities in which case the conductive heat fluxes can be finite with a zero temperature gradient. More precisely (see Jamet, 1998, pp. 213-230), the conditions which must be satisfied are:

$$Pe_v \ll 1$$

$$Pe_l \ll 1$$

$$\beta_v \frac{q_v L_v}{k_v} \ll 1$$

$$\beta_l \frac{q_l L_l}{k_l} \ll 1$$

where Pe is the Peclet number, β is the thermal expansion coefficient, L is a characteristic length and k is the thermal conductivity. Note that, for finite thermal conductivities, these conditions impose a limitation on the interfacial mass flux $\dot{m}^i$ through the interfacial energy balance equation.

3 Test-case description

In this section, we describe how the previous analytical results should be used as a numerical test. The problem is *one dimensional*, the length L of the domain being large enough so that a steady state in the framework linked to the interface can be achieved during the simulation. The main physical parameters of the problem are the *densities* of the bulk phases at saturation ρ_l and ρ_v. These parameters are the only ones for the incompressible model and the equations of state of the phases $P_l(\rho_l)$ and $P_v(\rho_v)$ are two other characteristics necessary for the compressible model. No restriction is imposed on any of these physical characteristics but they should be provided. For the incompressible model, viscosity does not play any role. For the compressible model, viscous effects should be weak compared to inertial effects (see equation (2)). The boundary conditions consist in imposing *constant and in general different mass fluxes* $\dot{m}_l$ *and* $\dot{m}_v$ on the liquid and vapor boundaries.

The equations solved should be, in general, *unsteady*, which means that *the interface position should vary in time*[2]. However, one should seek for a solution in which *the speed of displacement of the interface is constant.* The initial conditions are not imposed in this test-case; however, to decrease the unsteady stage of the computation (in terms of the speed of displacement of the interface), it is recommended to use the analytical solution given in section 2, namely equations (13)-(18) for an incompressible model and equations (23)-(28) for a compressible model.

The main numerical results that should be compared to the analytical solution are

1. the speed of displacement of the interface as a function of time $V^i(t)$ which must reach an asymptotic value given by equation (17) for an incompressible model and equation (27) for a compressible model;
2. the bulk pressures P_l and P_v and in particular their values at the interface P_l^i and P_v^i which should also reach asymptotic values given by equations (15) and (16) for an incompressible model and by equations (23) and (24) for a compressible model;
3. the interfacial mass flux $\dot{m}^i$;
4. the bulk velocities V_l and V_v and in particular their values at the interface V_l^i and V_v^i.

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[2] The only case for which the interface position is stationnary corresponds to the particular boundary conditions $\dot{m}_l = \dot{m}_v$.

It is recommended that the L^1-norm of the error (difference between the numerical results and the analytical results once the steady-state is reached) should be provided as a function of the mesh size and for all the above physical quantities.

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