Test-case number 27: Interface tracking based on an imposed velocity field in a convergent-divergent channel (PN)

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1 Practical significance and relevance of the test-case

The purpose of the problem described here is to assess interface tracking methods in incompressible, two-dimensional two-phase flow on the basis of an analytically given velocity field. Imposing an analytical velocity field has the advantage of simple and well-defined test conditions. In particular, the assessment of tracking methods is not affected by numerical aspects of Navier-Stokes solvers. On the other hand, imposed velocity fields are often poorly related to realistic experimental situations, as, for instance, the velocity fields proposed by Zalesak (1979) and B. & Kothe (1995). By contrast, the test problem proposed here can be considered as a reasonable approximation to the physical process of a bubble advected by laminar flow in a contracted channel.

A particular feature of the test problem is the curved boundary of the flow field. Solving the Navier-Stokes equation in this situation would require particular measures, such as introducing obstacles in a rectangular mesh configuration, or using cuvilinear coordinates. For an imposed velocity field, however, no particular measures are necessary to solve the problem, even with Cartesian coordinates. In fact, the detailed problem description given below is entirely based on Cartesian coordinates.

An important element of our test procedure is the construction of a discretely solenoidal velocity field, as an alternative to infinitesimally solenoidal fields used in the aforementioned references.

2 Definitions and model description

We consider a channel where the center line coincides with the x-axis of the coordinate system, and the function $f(x)$ specifies the (positive) distance between the channel wall and the center line (see Figure 1). The velocity field is derived from the assumption that the component u parallel to the center line has a parabolic profile in every cross-section of the channel. This assumption is strictly true in the case of laminar, stationary flow and straight channel walls, characterized by $f(x) =$ constant. For general channel shapes, the assumption can be expected to hold approximately, in particular if the variation of

$f(x)$ is weak.

The only solenoidal velocity field (u, v) having a parabolic profile of u in every cross-section of the channel is given – up to a constant factor – by the equations,

$$
\begin{aligned}
u &= \left(1 - \left(\frac{y}{f(x)}\right)^2\right) \frac{1}{f(x)} \,, \\
v &= \left(1 - \left(\frac{y}{f(x)}\right)^2\right) \frac{y}{(f(x))^2} \frac{\mathrm{d}}{\mathrm{d}x} f(x) \,.
\end{aligned}
\tag{1}
$$

Equations (1) apply, of course, to the region $|y| \leq f(x)$. Since the channel will be included in a rectangular computational domain (see Section 3), we extend the definition of u and v to the whole (x, y) plane by setting $u = 0$ and $v = 0$ in the region $|y| \geq f(x)$. The velocity field (u, v) defined in this way can be derived from the stream function

$$
\Psi = \begin{cases}
\frac{1}{3}\left(\frac{y}{f(x)}\right)^3 - \frac{y}{f(x)} & \text{if } |y| \leq f(x) \\
+\frac{2}{3} & \text{if } y \leq -f(x) \\
-\frac{2}{3} & \text{if } y \geq f(x)
\end{cases}
\tag{2}
$$

according to the equations,

$$
u = -\frac{\partial \Psi}{\partial y} \,, \quad v = \frac{\partial \Psi}{\partial x} \,. \tag{3}
$$

Thus the field (u, v) is solenoidal.

However, a solenoidal velocity field does not necessarily provide suitable test conditions, since some interface tracking algorithms require *discretely solenoidal* fields in order to work properly. For a discussion of this issue we restrict ourselves to a discretization scheme with square computational cells and face-centered velocity nodes. In this case, a velocity field (u, v) is discretely solenoidal if every computational cell satisfies the condition

$$
u_2 - u_1 + v_2 - v_1 = 0, \tag{4}
$$

where u_1, u_2, v_1, v_2 are velocity components on the face centers as indicated in figure 2. However, if u_1, u_2, v_1, v_2 are derived from (3), we get (from a Taylor expansion around the cell center)

$$
u_2 - u_1 + v_2 - v_1 = \left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 v}{\partial y^3}\right)_{\mathrm{cc}} \frac{h^3}{24} + O(h^5) \tag{5}
$$

where h is the side length of computational cells, and the subscript “cc” refers to the cell center. We have actually identified the non-vanishing right-hand side in (5) as the reason for numerical artifacts in some VOF tests, as discussed below in Section 3.

To associate a stream function Ψ with a discretely solenoidal velocity field, we propose to use face-centered velocities defined by,

$$
\begin{aligned}
u_1 &= -\frac{\Psi_{12} - \Psi_{11}}{h} \ , \quad u_2 = -\frac{\Psi_{22} - \Psi_{21}}{h} \ , \\
v_1 &= \frac{\Psi_{21} - \Psi_{11}}{h} \ , \quad v_2 = \frac{\Psi_{22} - \Psi_{12}}{h} \ ,
\end{aligned} \tag{6}
$$

where Ψ_{11}, Ψ_{12}, Ψ_{21}, Ψ_{22} are the values of Ψ in the corners of the corresponding computational cell, as indicated in figure 2. The velocities defined by (6) satisfy (4) exactly. They differ from their infinitesimally solenoidal counterparts derived from (3) by a term of order h^2.

3 Test-case description

It is proposed to track the interface of a bubble using the following parameters:

- *Channel shape.* The channel profile is given by

$$
f(x) = 1 - a \exp\left[-\frac{1}{2}\left(\frac{x}{b}\right)^2\right] \tag{7}
$$

  with shape parameters $a = 0.75$ and $b = 0.5$. The maximum velocity resulting from (1) and (7) is

$$
w_{\max} \equiv \max \sqrt{u^2 + v^2} = \frac{1}{f(0)} = \frac{1}{1-a} = 4 \ . \tag{8}
$$

  The maximum occurs at the point $(0, 0)$.

- *Computational domain; discretization of space.* The computational domain is the rectangle $[-2.5, 2.5] \times [-1.25, 1.25]$ divided into 256×128 computational cells, implying square computational cells of side length $h = 0.01953125$.

- *Initial state.* In the initial state, at time $t = 0$, the bubble is circular. The center is located at the point $(-1.95, 0)$, and the radius equals 0.5.

- *Final states; discretization of time.* Two final states are considered: at time $t = t_1 = 1220\Delta t$, and at $t = t_2 = 2020\Delta t$. The time step Δt corresponds to the Courant number $C = 0.3$. This implies

$$
\Delta t = \frac{hC}{w_{\max}} = 0.00146484375 \ . \tag{9}
$$

It is required to compare the interface of the two final states with "exact" solutions that are obtained by means of advected marker points. Marker point advection is governed by the ordinary differential equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}(x, y) = (u, v) \tag{10}
$$

with (u, v) given by (1). The numerical error of the corresponding solution can easily be controlled, which justifies considering marker point solutions as an exact reference.

Figure 1 shows the initial state of the bubble as well as the exact final states. The advection of 13 selected marker points on the interface (equally spaced in the initial state) is also shown. The coordinates of the marker points are listed in table 1. The error of the tabulated data is less than one unit of the last decimal place given.

Figure 3 shows the result of the test procedure when applied to the following two interface tracking methods: (i) unsplit PLIC VOF (Meier *et al.* , 2000), and (ii) Level Set (Sussman *et al.* , 199). The results illustrate a characteristic complementarity between VOF and Level Set methods: VOF methods have superior volume conservation properties, but are liable to develop small-scale topological irregularities like the outward bend of the two “fin tips” in the state $t = t_2$.

The VOF results presented in figure 3 are based on discretely solenoidal fields, with face-centered velocities derived from (6). When face-centered velocities are derived instead from (1), we obtain spurious contaminations of one phase by the other, as well as some additional small-scale irregularities of the interface. This finding indicates that discretely solenoidal velocity fields can be a prerequisite for obtaining meaningful test results.

i	$t = 0$		$t = t_1$		$t = t_2$	
	x_i	y_i	x_i	y_i	x_i	y_i
1	-1.450	0.000	1.270	0.000	2.447	0.000
2	-1.467	0.129	1.221	0.126	2.380	0.131
3	-1.517	0.250	1.081	0.234	2.193	0.252
4	-1.596	0.354	0.865	0.296	1.929	0.355
5	-1.700	0.433	0.571	0.264	1.641	0.432
6	-1.821	0.483	0.074	0.125	1.384	0.476
7	-1.950	0.500	-0.423	0.238	1.201	0.479
8	-2.079	0.483	-0.602	0.307	1.117	0.453
9	-2.200	0.433	-0.660	0.297	1.133	0.408
10	-2.304	0.354	-0.648	0.239	1.219	0.340
11	-2.383	0.250	-0.599	0.159	1.327	0.244
12	-2.433	0.129	-0.547	0.076	1.414	0.128
13	-2.450	0.000	-0.525	0.000	1.447	0.000

Table 1: Coordinates of selected marker points on the interface in the initial state ($t = 0$) and the two final states considered in the test problem ($t = t_1$, and $t = t_2$).

References

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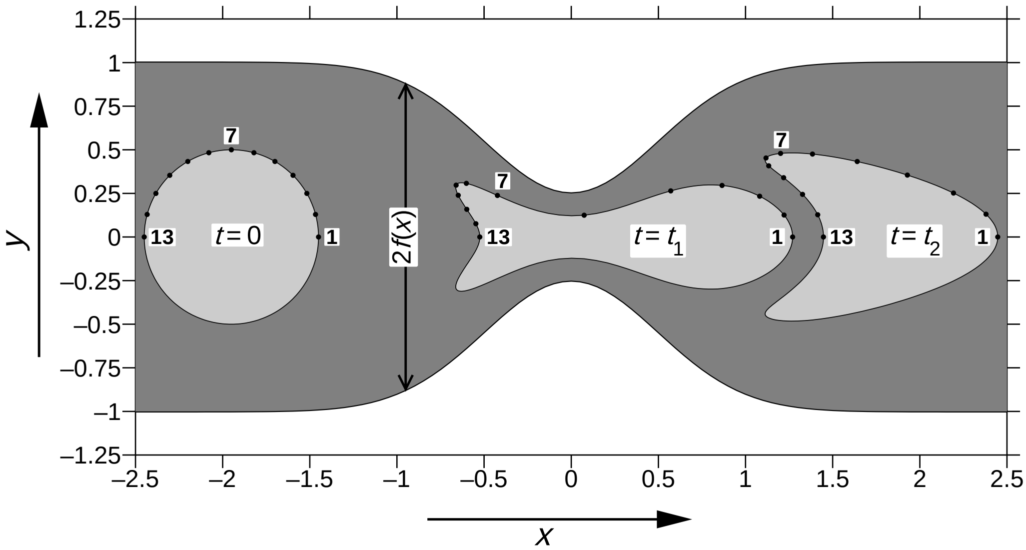

Figure 1: Computational domain, with channel region (dark grey), channel width $2f(x)$, and the three states of the advected bubble (bright grey) considered in the test problem. The black dots are marker points advected by the flow. The labels 1, 7, and 13 refer to the numbering scheme used in table 1.

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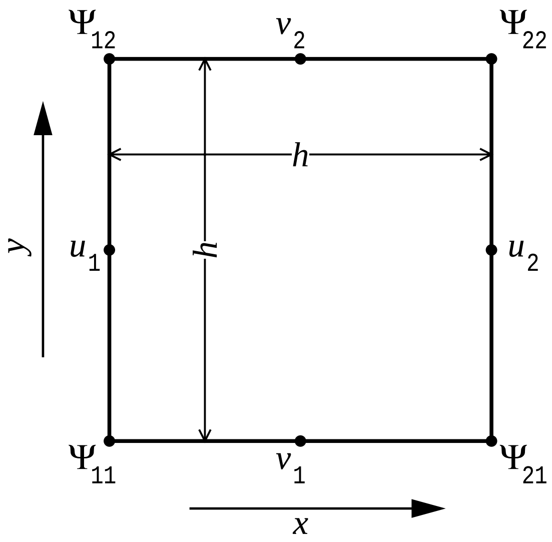

Figure 2: Computational cell of size h, with stream function values $\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22}$ in the corners, and velocity components u_1, u_2, v_1, v_2 at face centers.

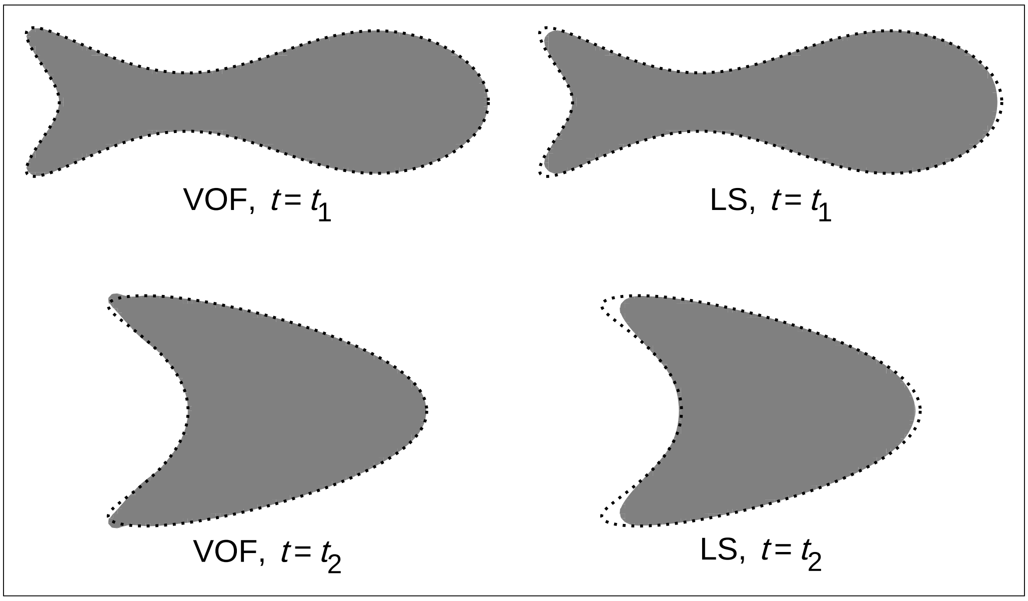

Figure 3: The grey area represents final bubble shapes obtained with two different interface tracking methods, Volume of Fluid (VOF) and Level Set (LS). The dotted lines indicate exact interface curves obtained by means of marker point advection.