Test-case number 33: Propagation of solitary waves in constant depths over horizontal beds (PA, PN, PE)

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1 Practical significance and interest of the test-case

Analytical solutions and a related experiment are provided here. The experiment is devoted to the propagation of a solitary wave in a wave tank. In particular, the water surface profiles and corresponding water particle velocities of several solitary waves have been measured. These experimental results can be compared with existing analytical theories which follow different orders of approximation. Solitary waves are known for having some interesting properties: indeed, such a wave has a symmetrical form with a single hump and propagates with a uniform velocity without changing form.

When simulating two-phase flows, it is important to evaluate the general accuracy of the numerical methods and numerical schemes used by checking for example the balance of mass and energy in the computing domain. Thus, the results of the different solitary wave theories can be used to compute the initial kinematic properties and simulate their propagation in constant depths over horizontal beds in periodic domains, the precision of the simulation being assessed by comparing the free-surface shapes and velocities to the theoretical values.

2 Definitions and model description

Several analytical solutions can be found in the literature (Boussinesq, 1871, McCowan, 1891, Grimshaw, 1971, Fenton, 1972, Lee *et al.* , 1982). The reference variables of the initial wave are the celerity, c, the depth, d, of the water channel and the wave height, H. All the solutions detailed in the following sections will give solitary waves propagating from the left to the right end of the numerical domain.

Boussinesq solution or 1^{st} order solution

The initial wave shape, and the shape at any time t, celerity and velocity field (u and v being the cartesian components) can be computed from the first order solitary wave theory (Boussinesq, 1871):

$$\eta(x,t) = H\text{sech}^2\Big(\sqrt{\frac{3H}{4d^3}}(x - x_0 - ct)\Big), \tag{1}$$

where x_0 is the initial position of the wave crest. The celerity is defined by:

$$c = \sqrt{g(d+H)}. \tag{2}$$

In the following, it will be considered that η is a sole function of the coordinate x' which is the longitudinal coordinate in the frame attached to the wave.

$$\eta = \eta(x'(x,t)) \text{ with } x'(x,t) = x - x_0 - ct. \tag{3}$$

Next the velocity field is given by:

$$\begin{aligned} \frac{u}{\sqrt{gd}} &= \epsilon\Big[\eta_* - \frac{1}{4}\epsilon{\eta_*}^2 + \frac{d^2}{3c^2}\Big[1 - \frac{3}{2}\frac{z^2}{d^2}\Big]\frac{\partial^2\eta_*}{\partial t^2}\Big] \\ \frac{v}{\sqrt{gd}} &= z\frac{\epsilon}{c}\Big[\Big(1 - \frac{1}{2}\epsilon\eta_*\Big)\frac{\partial\eta_*}{\partial t} + \frac{d^2}{3c^2}\Big(1 - \frac{z^2}{2d^2}\Big)\frac{\partial^3\eta_*}{\partial t^3}\Big] \end{aligned} \tag{4}$$

with $\epsilon = H/d$, $\eta_* = \eta/H$, and:

$$\begin{aligned} \frac{\partial\eta_*}{\partial t} &= \frac{2c\sqrt{\frac{3}{4}\frac{H}{d^3}}\sinh\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right)}{\cosh^3\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right)} \\ \frac{\partial^2\eta_*}{\partial t^2} &= \frac{c^2\frac{3}{2}\frac{H}{d^3}\left[2\cosh^2\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right) - 3\right]}{\cosh^4\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right)} \\ \frac{\partial^3\eta_*}{\partial t^3} &= \frac{6c^3\frac{H}{d^3}\sqrt{\frac{3}{4}\frac{H}{d^3}}\sinh\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right)\left[\cosh^2\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right) - 3\right]}{\cosh^5\left(\sqrt{\frac{3}{4}\frac{H}{d^3}}x'\right)} \end{aligned} \tag{5}$$

A lower order solution can be given:

$$\begin{aligned} \frac{u}{\sqrt{gd}} &= \frac{\eta}{d} \\ \frac{v}{\sqrt{gd}} &= \frac{z}{d}\frac{1}{c}\frac{\partial\eta}{\partial t} \end{aligned} \tag{6}$$

Note that this solution is a very rough one as the horizontal velocity component, u, is independent of depth, z.

Grimshaw solution or 3^{rd} order solution

The initial wave shape, celerity and velocity field are then given by the following solution developed by Grimshaw (1971):

$$\frac{\eta}{d} = \epsilon s^2 - \frac{3}{4}\epsilon^2 s^2 t^2 + \epsilon^3\Big(\frac{5}{8}s^2t^2 - \frac{101}{80}s^4t^2\Big) \tag{7}$$

$$
\begin{aligned}
&\frac{u}{\sqrt{gd}} = \epsilon s^2 - \epsilon^2\Big[-\frac{1}{4}s^2 + s^4 + \Big(\frac{z}{d}\Big)^2\Big(\frac{3}{2}s^2 - \frac{9}{4}s^4\Big)\Big] \\
&-\epsilon^3\Big[\frac{19}{4}s^2 + \frac{1}{5}s^4 - \frac{6}{5}s^6 + \Big(\frac{z}{d}\Big)^2\Big(-\frac{3}{2}s^2 - \frac{15}{4}s^4 + \frac{15}{2}s^6\Big) \\
&+\Big(\frac{z}{d}\Big)^4\Big(-\frac{3}{8}s^2 + \frac{45}{16}s^4 - \frac{45}{16}s^6\Big)\Big] \\
&\frac{v}{\sqrt{gd}} = (3\epsilon)^{\frac{1}{2}}\Big(\frac{z}{d}\Big)t\Big[-\epsilon s^2 + \epsilon^2\Big[\frac{3}{8}s^2 + 2s^4 + \Big(\frac{z}{d}\Big)^2\Big(\frac{31}{2}s^2 - \frac{3}{2}s^4\Big)\Big] \\
&+\epsilon^3\Big[\frac{49}{640}s^2 - \frac{17}{20}s^4 - \frac{18}{5}s^6 + \Big(\frac{z}{d}\Big)^2\Big(-\frac{13}{16}s^2 - \frac{25}{16}s^4 + \frac{15}{2}s^6\Big) \\
&+\Big(\frac{z}{d}\Big)^4\Big(-\frac{3}{40}s^2 + \frac{9}{8}s^4 - \frac{27}{16}s^6\Big)\Big]\Big]
\end{aligned} \tag{8}
$$

with $\epsilon = H/d$, $\alpha = (\frac{3}{4}\epsilon)^{\frac{1}{2}}(1 - \frac{5}{8}\epsilon + \frac{71}{128}\epsilon^2)$, $s = \text{sech}(\alpha x') = 1/\cosh(\alpha x')$ et $t = \tanh(\alpha x')$.

The pressure and celerity are then given by:

$$
\begin{aligned}
&\frac{p}{\rho g d} = 1 - \Big(\frac{z}{d}\Big) + \epsilon s^2 + \epsilon^2\Big[\frac{3}{4}s^2 - \frac{3}{2}s^4 + \Big(\frac{z}{d}\Big)^2\Big(-\frac{3}{2}s^2 + \frac{9}{4}s^4\Big)\Big] \\
&+\epsilon^3\Big[-\frac{1}{2}s^2 - \frac{19}{20}s^4 + \frac{11}{5}s^6 \\
&+\Big(\frac{z}{d}\Big)^2\Big(\frac{3}{4}s^2 + \frac{39}{8}s^4 - \frac{33}{4}s^6\Big) + \Big(\frac{z}{d}\Big)^4\Big(\frac{3}{8}s^2 - \frac{45}{16}s^4 + \frac{45}{16}s^6\Big)\Big]
\end{aligned} \tag{9}
$$

$$
c = \sqrt{gd}\Big(1 + \epsilon - \frac{1}{20}\epsilon^2 - \frac{3}{70}\epsilon^3\Big)^{\frac{1}{2}} \tag{10}
$$

Some other solutions can be found in the literature, as McCowan's analytical solution (McCowan, 1891), Fenton's 9^{th} order solution (Fenton, 1972) or Tanaka's exact solution (Tanaka, 1986), the latter two being numerically obtained.

3 A series of three test-cases

It is proposed to simulate the propagation of solitary waves in three different configurations, according to the parameters of table 1. Water particle velocities for various depths and wave profiles are measured by Lee *et al.* (1982) along the time, as shown in figures 1, 2, 3, 4 and 5. Results obtained with Tanaka's algorithm (Tanaka, 1986) are also compared.

As described in section 2, the solitary wave is completely defined for a given depth, d, and relative amplitude, ϵ. Therefore, it is possible to compare the computed evolution of the wave profiles and velocities during the non-dimensional time $t\sqrt{g/d}$, at various depths, z, to the results of the experiments and to the analytical solutions for the three values of ϵ.

Test-case	ϵ	$d(m)$	H (m)
1	0.11	0.302	0.03322
2	0.19	0.4046	0.076874
3	0.29	0.204	0.05916

Table 1: Values of the parameters describing the experiments by Lee *et al.* (1982), with $\epsilon = H/d$.

The proposed numerical configuration is to consider an initial solitary wave computed from a chosen analytical theory, any of those presented in section 2 giving good results, except the already mentioned low order solution, Eq. 6. The free-surface shapes are almost identical, as shown in figure 1, whatever the analytical theory. The differences between the analytical theories can be estimated from the tables 2, 3 and 4, where we give sample of the extremal value of the non-dimensional cartesian components of the velocities for given depths. The most appropriate case for an accurate comparison is the test-case 1, with $d = 0.3020\ m$ and $\epsilon = H/d = 0.11$, the prediction of the wave profile being guaranteed to less than 1.10^{-3}m.

The crest is located in the middle of the numerical domain, periodic boundary conditions being imposed in the flow direction. All calculations should be made with the densities and the viscosities of air and water ($\rho_a = 1.1768$ kg.m^{-3} and $\rho_w = 1000$ kg.m^{-3}, $\mu_a = 1.85.10^{-5}$ kg.m^{-1}.s^{-1} and $\mu_w = 1.10^{-3}$ kg.m^{-1}.s^{-1}). It is obvious that a sufficient number of grid points may be chosen in order to have enough accuracy in the free-surface description. The simulation time step is chosen to verify the stability criterion (Courant-Friedrichs-Levy) less than one for the interface algorithm.

It is required to check that the solitary wave maintains its original shape as it propagates. The differences can be calculated between the theoretical and numerical results. The free-surface profiles are to be plotted versus the non-dimensional time, as shown in figure 1, and compared to the analytical values with 1 and 7. It is also required to plot the velocity distributions along the depth, as the wave propagates, versus the non-dimensional time, as shown in figures 2, 3, 4 and 5. For an easier comparison, the main values to be checked are given in tables 2, 3 and 4.

The analytical models are non dissipative. Therefore, the conservation of mass and energy may be checked during the numerical simulation, the kinetic energy, the potential energy and the total energy being:

$$
\begin{aligned}
E_k &= \frac{1}{2}\int\int \rho u^2 dxdy \\
E_p &= \int\int \rho z dxdy \\
E_t &= E_k + E_p
\end{aligned}
\tag{11}
$$

As a matter of fact, an easy check to do is to consider the celerity of the initial wave, estimate the theoretical distance it has to propagate during the time of the simulation and compare it to the final position of the wave crest.

Summary of the required calculations for propagations of solitary waves

Three cases of solitary waves propagating in constant depths over horizontal beds are proposed. The test case control parameters are given in table 1, an initial analytical

1^{st} order				
c (m.s^{-1})	1.8134			
$c/\sqrt{gd}$	1.054			
z/d	0.92	0.78	0.62	0.45
$u_{max}/\sqrt{gd}$	0.109	0.106	0.104	0.103
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.202	± 0.170	± 0.133	± 0.0958
3^{rd} order				
c (m.s^{-1})	1.81288			
$c/\sqrt{gd}$	1.053			
z/d	0.92	0.78	0.62	0.45
$u_{max}/\sqrt{gd}$	0.107	0.106	0.104	0.103
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.195	± 0.163	± 0.129	± 0.0927
Tanaka				
c (m.s^{-1})	1.81284			
$c/\sqrt{gd}$	1.053			
z/d	0.92	0.78	0.62	0.45
$u_{max}/\sqrt{gd}$	0.106	0.105	0.103	0.102
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.195	± 0.165	± 0.128	± 0.0952

Table 2: Values of the maximum of the non-dimensional velocity components at given non-dimensional depths (z/d). Test case 1, $d = 0.3020$ m, $\epsilon = 0.11$.

	1^{st} order			
c (m.s^{-1})	2.1733			
z/d	1.05	1.02	0.92	0.45
$u_{max}/\sqrt{gd}$	0.193	0.191	0.186	0.168
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.500	± 0.484	± 0.429	± 0.199
	3^{rd} order			
c (m.s^{-1})	2.1714			
z/d	1.05	1.02	0.92	0.45
$u_{max}/\sqrt{gd}$	0.182	0.181	0.178	0.169
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.465	± 0.450	± 0.402	± 0.189
	Tanaka			
c (m.s^{-1})	2.1713			
z/d	1.05	1.02	0.92	0.45
$u_{max}/\sqrt{gd}$	0.184	0.183	0.179	0.170
$v_{max}/\sqrt{gd}$ (.10^{-1})	± 0.469	± 0.457	± 0.408	± 0.198

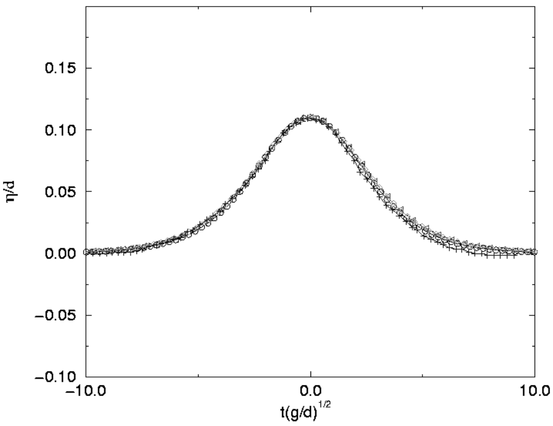
Table 3: Values of the maximum of the non-dimensional velocity components at given non-dimensional depths (z/d). Test case 2, $d = 0.4046$ m, $\epsilon = 0.19$.

theory has to be chosen between those presented in 2. The two-phase flow should be simulated with the densities and the viscosities of air and water ($\rho_a = 1.1768$ kg.m^{-3} and $\rho_w = 1000$ kg.m^{-3}, $\mu_a = 1.85.10^{-5}$ kg.m^{-1}.s^{-1} and $\mu_w = 1.10^{-3}$ kg.m^{-1}.s^{-1}). It is required to check the following results with the reference model:

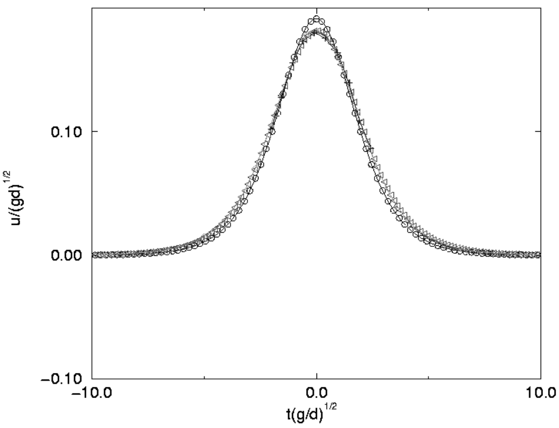
- The conservation of the shapes of the solitary waves.
- The conservation of mass and energy during the simulation.
- The total distance of propagation during the simulation.

	1^{st} order			
c (m.s^{-1})	1.6067			
z/d	1.05	1.03	0.92	0.67
$u_{max}/\sqrt{gd}$	0.296	0.294	0.280	0.255
$v_{max}/\sqrt{gd}$ (.10^{-1})	$\pm$ 0.876	$\pm$ 0.855	$\pm$ 0.745	$\pm$ 0.522
	3^{rd} order			
c (m.s^{-1})	1.6035			
z/d	1.05	1.03	0.92	0.67
$u_{max}/\sqrt{gd}$	0.260	0.259	0.253	0.245
$v_{max}/\sqrt{gd}$ (.10^{-1})	$\pm$ 0.786	$\pm$ 0.770	$\pm$ 0.680	$\pm$ 0.485
	Tanaka			
c (m.s^{-1})	1.6032			
z/d	1.05	1.03	0.92	0.67
$u_{max}/\sqrt{gd}$	0.273	0.271	0.264	0.251
$v_{max}/\sqrt{gd}$(.10^{-1})	$\pm$ 0.788	$\pm$ 0.770	$\pm$ 0.672	$\pm$ 0.471

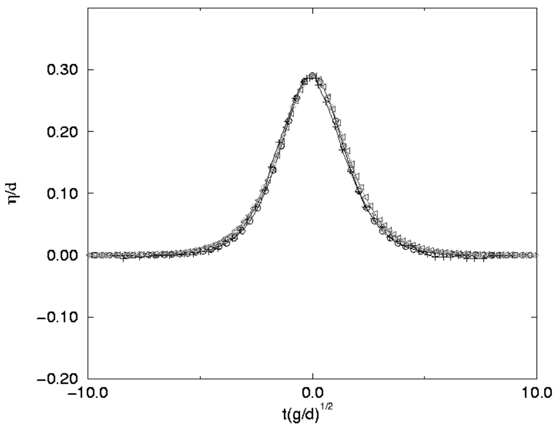
Table 4: Values of the maximum of the non-dimensional velocity components at given non-dimensional depths (z/d). Test case 3, $d = 0.204$ m, $\epsilon = 0.29$.



a. $d = 0.302$ m, $\epsilon = 0.11$

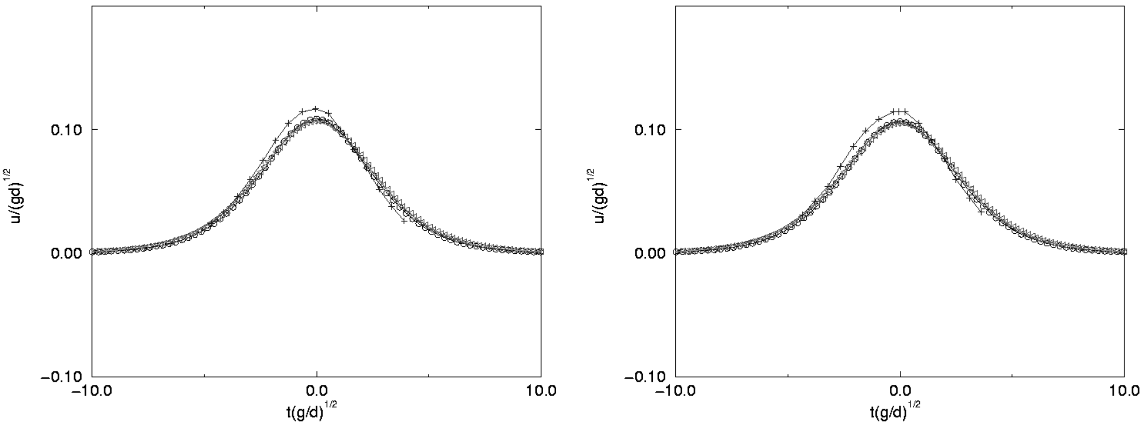


b. $d = 0.4046$ m, $\epsilon = 0.19$

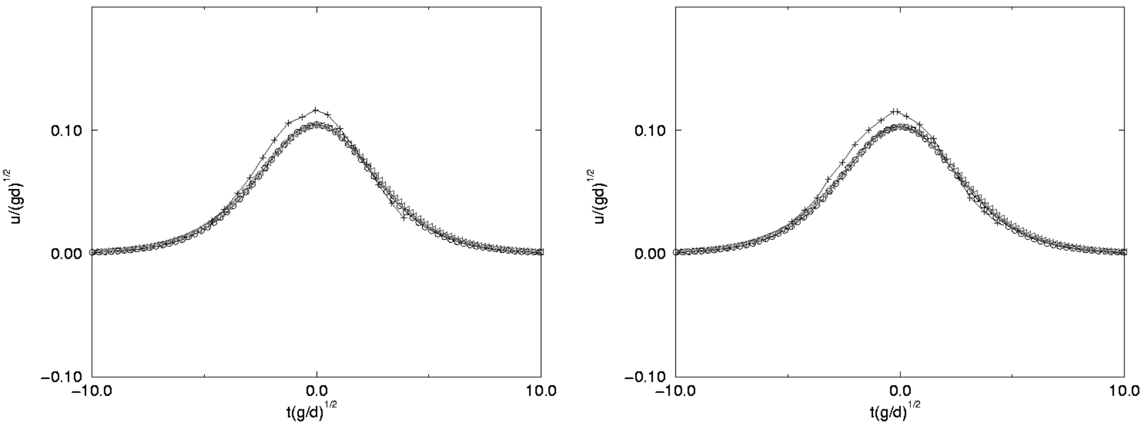


c. $d = 0.204$ m, $\epsilon = 0.29$

Figure 1: Solitary wave profiles: water surface elevation, η/d, is plotted versus non-dimensional time, $t\sqrt{g/d}$. Comparison of theories and experiments. $\circ$ 1^{st} order, $\triangleleft$ 3^{rd} order, $\diamond$ Tanaka (Tanaka, 1986), $+$ Lee (Lee *et al.* , 1982).

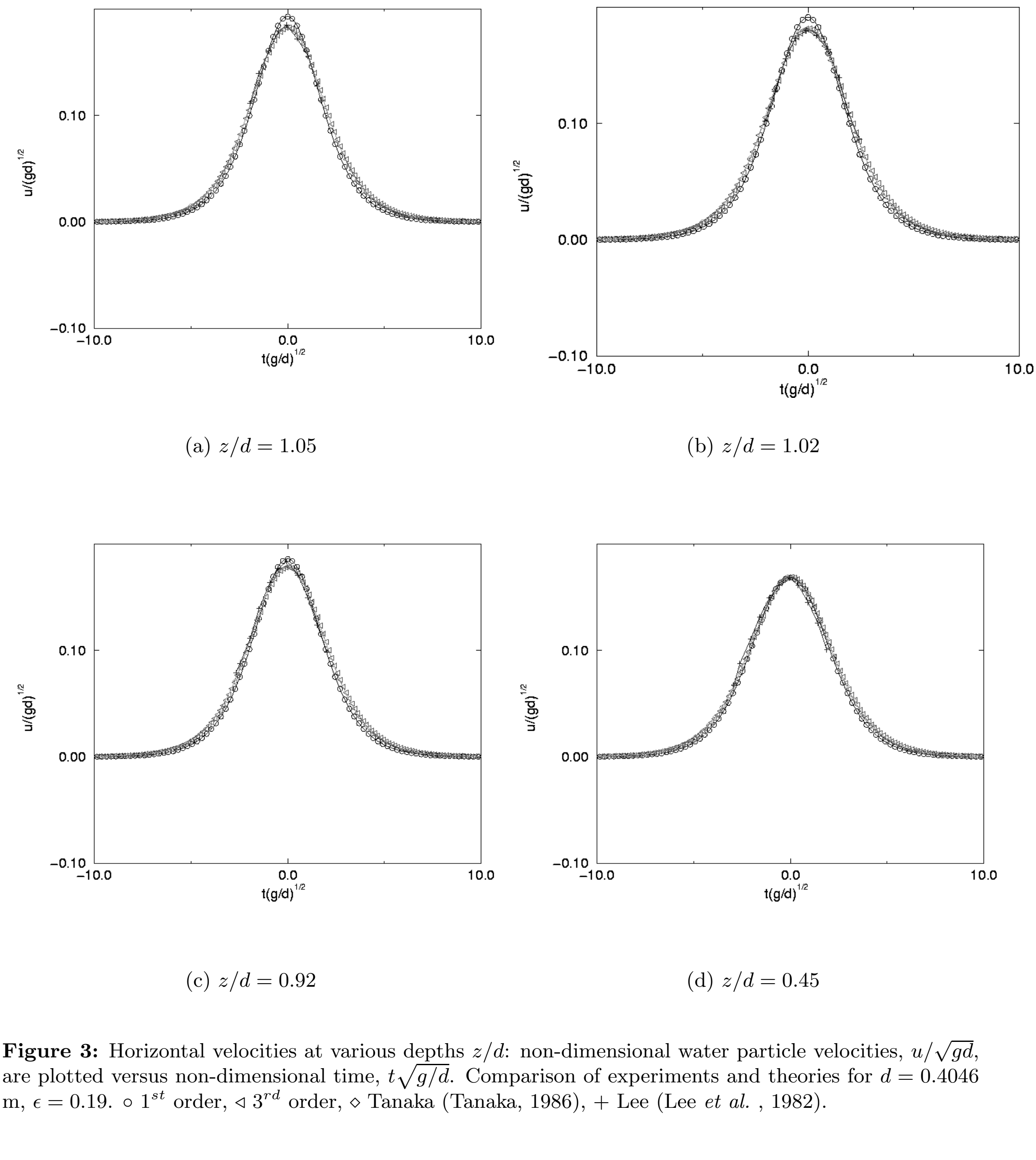


(a) $z/d = 0.92$ (b) $z/d = 0.78$



(c) $z/d = 0.62$ (d) $z/d = 0.45$

Figure 2: Horizontal velocities at various depths z/d: non-dimensional water particle velocities, $u/\sqrt{gd}$, are plotted versus non-dimensional time, $t\sqrt{g/d}$. Comparison of experiments and theories for $d = 0.3020$ m, $\epsilon = 0.11$. $\circ$ 1^{st} order, $\triangleleft$ 3^{rd} order, $\diamond$ Tanaka (Tanaka, 1986), + Lee (Lee *et al.* , 1982).



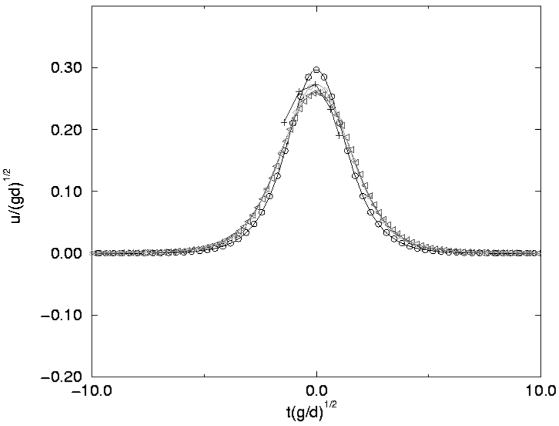
(a) $z/d = 1.05$

(b) $z/d = 1.02$

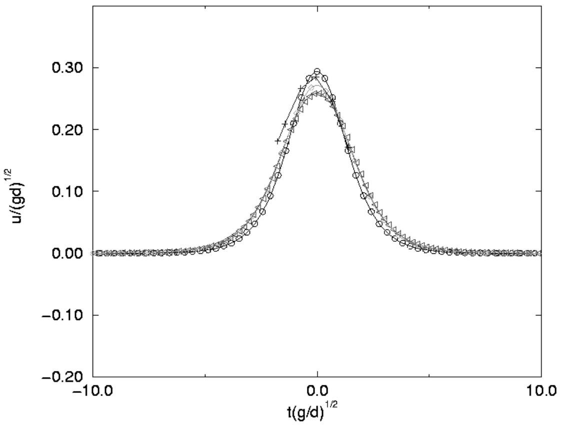
(c) $z/d = 0.92$

(d) $z/d = 0.45$

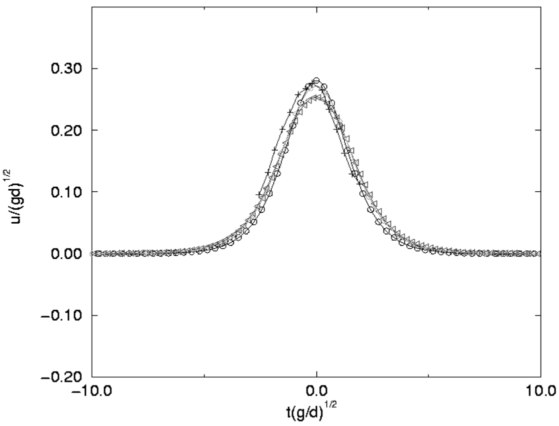
Figure 3: Horizontal velocities at various depths z/d: non-dimensional water particle velocities, $u/\sqrt{gd}$, are plotted versus non-dimensional time, $t\sqrt{g/d}$. Comparison of experiments and theories for $d = 0.4046$ m, $\epsilon = 0.19$. $\circ$ 1^{st} order, $\triangleleft$ 3^{rd} order, $\diamond$ Tanaka (Tanaka, 1986), + Lee (Lee *et al.* , 1982).



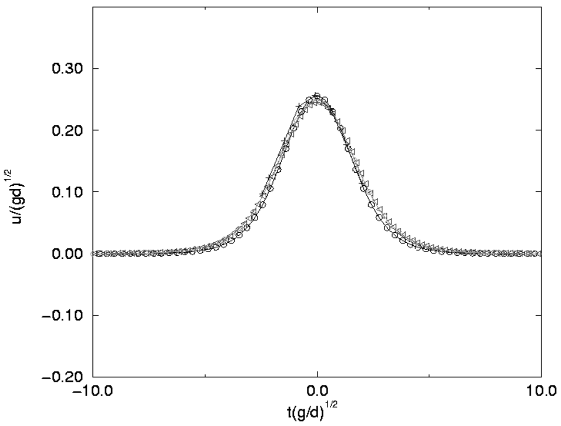
(a) $z/d = 1.05$



(b) $z/d = 1.03$

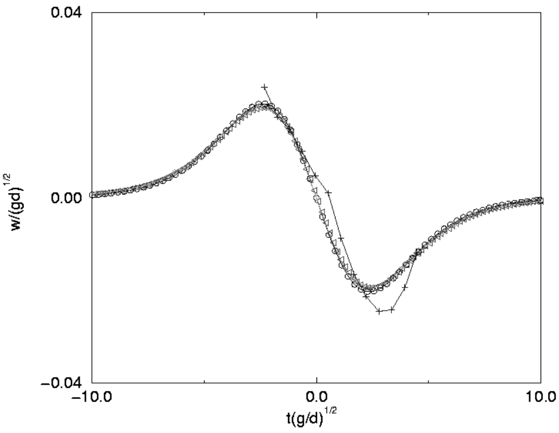


(c) $z/d = 0.92$

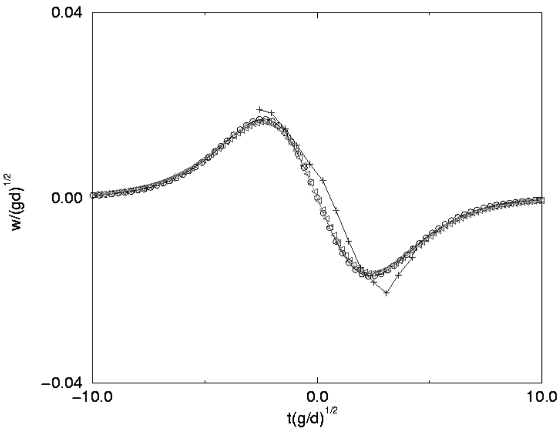


(d) $z/d = 0.67$

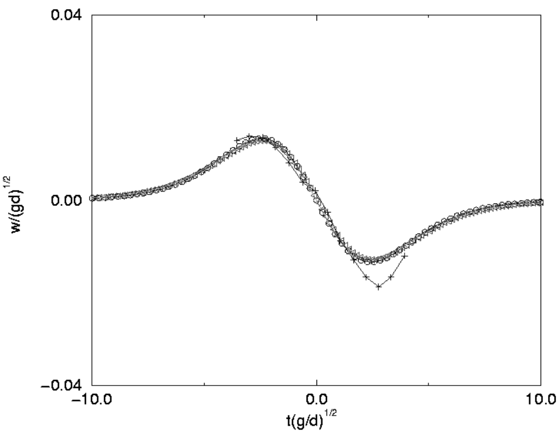
Figure 4: Horizontal velocities at various depths z/d: non-dimensional water particle velocities, $u/\sqrt{gd}$, are plotted versus non-dimensional time, $t\sqrt{g/d}$. Comparison of experiments and theories for $d = 0.204$ m, $\epsilon = 0.29$. $\circ$ 1^{st} order, $\triangleleft$ 3^{rd} order, $\diamond$ Tanaka (Tanaka, 1986), + Lee (Lee *et al.* , 1982).



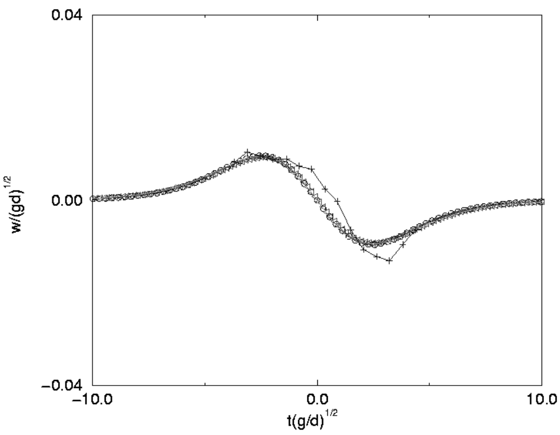
(a) $z/d = 0.92$



(b) $z/d = 0.78$



(c) $z/d = 0.62$



(d) $z/d = 0.45$

Figure 5: Vertical velocities at various depths z/d: non-dimensional water particle velocities, $v/\sqrt{gd}$, are plotted versus non-dimensional time, $t\sqrt{g/d}$. Comparison of experiments and theories for $d = 0.302$ m, $\epsilon = 0.11$. $\circ$ 1^{st} order, $\triangleleft$ 3^{rd} order, $\diamond$ Tanaka (Tanaka, 1986), + Lee (Lee *et al.* , 1982).

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