Test-case number 34: Two-dimensional sloshing in cavity - an exact solution (PA)

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1 Practical significance and interest of the test-case

This test case is an analytical one. We provide an exact solution for the flow in a rectangular cavity of two non-miscible inviscid fluids of different densities ρ_1 and ρ_2 with $\rho_2 > \rho_1$. The two layers of fluid are superimposed with the lighter one over the heavier one. Gravity is acting in the vertical downward direction. The description of the configuration is given in figure 1. At initial time, both fluids are at rest, then the cavity is submitted to an horizontal time dependant acceleration noted $a(t)$.

2 Definitions and physical model description

2.1 Assumptions and model equations

We suppose that $a(t)$ is small enough such that fluid oscillations remain in the linear regime. The two fluids are also supposed to be incompressible, and inertial terms associated with the Bernoulli equation $(1/2v^2)$ are neglected compared to unsteady terms $(\frac{\partial\varphi}{\partial t})$. Indeed, those assumptions imply that the amplitude of the oscillations are small compared to the wavelength of the traveling wave. One can then assume that the flow of both fluids is potential. Let φ_1 and φ_2 be the potentials corresponding to fluid 1 and 2 respectively. Integration of the Euler equations for each phase implies:

$$\left\{\begin{array}{lcl} p_1 & = & -\rho_1 g y + \rho_1 a(t) x - \rho_1 \dfrac{\partial\varphi_1}{\partial t} \\ p_2 & = & -\rho_2 g y + \rho_2 a(t) x - \rho_2 \dfrac{\partial\varphi_2}{\partial t} \end{array}\right. \tag{1}$$

The mass balance equations for each fluid are given by:

$$\left\{\begin{array}{lcl} \Delta\varphi_1 & = & 0 \\ \Delta\varphi_2 & = & 0 \end{array}\right. \tag{2}$$

Since both fluids are inviscid, boundary conditions express that there is no flow across the wall by:

$$\left\{\begin{array}{lcl} \left(\dfrac{\partial\varphi_i}{\partial x}\right)_{x=0,L} & = & 0 \\ \\ \left(\dfrac{\partial\varphi_i}{\partial y}\right)_{y=-h_2,h_1} & = & 0 \end{array}\right. \tag{3}$$

where L is the length of the domain as shown in figure 1.

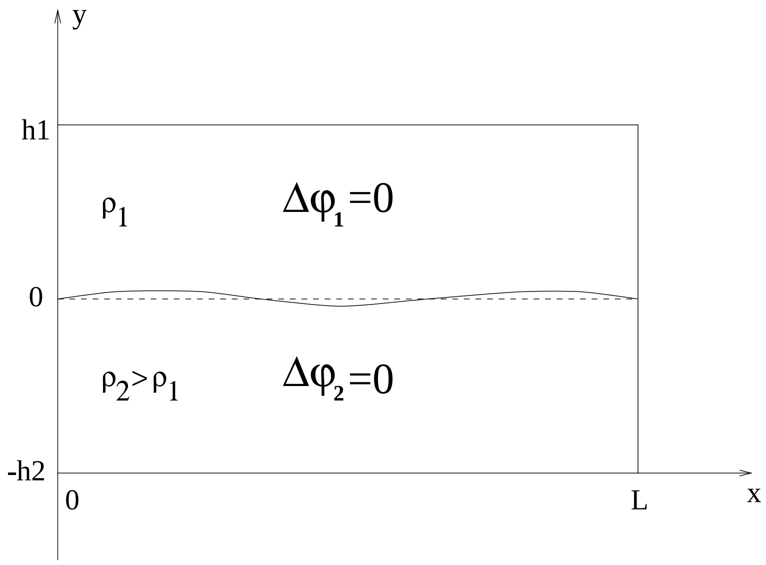

Figure 1: Problem definition. Gravity is vertical downward and the acceleration $a(t)$ is along the x axis.

2.2 Interface boundary conditions

Now, we have to specify interface boundary conditions. It is assumed that the effect of surface tension can be neglected. Therefore, the pressure is continuous across the interface. As a result of the mass balance at the interface, since there is no phase change, the normal component of the velocity at the interface must also be continuous.

If ξ denotes the vertical deviation of the interface from $y = 0$, for small amplitude oscillations and by linearizing he boundary conditions at the interface, one gets the following equations for the continuity of the pressure and the normal component of the velocity at the interface:

$$g\xi - ax + \frac{\rho_2}{\rho_2 - \rho_1}\frac{\partial \varphi_2}{\partial t} - \frac{\rho_1}{\rho_2 - \rho_1}\frac{\partial \varphi_1}{\partial t} = 0 \tag{4}$$

$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial \xi}{\partial t} = \frac{\partial \varphi_2}{\partial y} \tag{5}$$

From (4) and (5), after some algebra, one gets the following equation for ξ:

$$g\frac{\partial \xi}{\partial t} - a'x + \frac{\rho_2}{\rho_2 - \rho_1}\frac{\partial^2 \varphi_2}{\partial t^2} - \frac{\rho_1}{\rho_2 - \rho_1}\frac{\partial^2 \varphi_1}{\partial t^2} = 0 \tag{6}$$

2.3 Exact solutions

We seek for the solutions of (2) by the method of separation of variables in the following form:

$$\begin{aligned} \varphi_1 &= \sum_n \phi_n^1(t)\cosh(k_n(y - h_1))\cos(k_n x) \\ \varphi_2 &= \sum_n \phi_n^2(t)\cosh(k_n(y + h_2))\cos(k_n x) \end{aligned} \tag{7}$$

where the wave number, k_n, is defined by,

$$k_n = \frac{2\pi n}{L} \tag{8}$$

The interface conditions (5) et (6) give ϕ_n^i. By replacing in equation (6) $\frac{\partial\xi}{\partial t}$ respectively by $\frac{\partial\varphi_1}{\partial y}$ and $\frac{\partial\varphi_2}{\partial y}$ and projecting on the base $\cos(k_n x)$, one obtains finally the following system of ODE:

$$\left\{\begin{array}{lcl} \ddot{\phi}_n^1 + \omega_n^2\phi_n^1 &=& -\dfrac{a' X_n \omega_n^2}{g k_n \sinh k_n h_1} \\ \\ \ddot{\phi}_n^2 + \omega_n^2\phi_n^2 &=& +\dfrac{a' X_n \omega_n^2}{g k_n \sinh k_n h_2} \end{array}\right. \tag{9}$$

where X_n is the projection of x on the base $\cos(k_n x)$ and where the dot denotes the time derivative. To close the problem, we have to calculate the X_n. Those coefficients are given by:

$$\begin{array}{lcl} X_0 &=& \dfrac{L}{2} \\ \\ X_{2n} &=& 0 \\ \\ X_{2n+1} &=& -\dfrac{4L}{(2n+1)^2\pi^2} = -\dfrac{4}{L k_{2n+1}^2} \end{array} \tag{10}$$

In (9) ω_n is given by:

$$\omega_n^2 = \frac{g k_n \Delta\rho}{\rho_1 \coth(k_n h_1) + \rho_2 \coth(k_n h_2)} \tag{11}$$

with $\Delta\rho = \rho_2 - \rho_1$. This last relation is the dispersion equation of the problem. One can clearly see that as h_1 goes to infinity and if $\rho_1 \ll \rho_2$, the general dispersion relation for gravitational surface waves of depth h_2 is recovered: $\omega^2 = gk \tanh kh_2$. To solve the system of ODE (9), one needs initial conditions for ϕ_n^i. Since potentials are defined to an additive constant, one can choose $\phi_n^i(0) = 0$ with no loss of generality. Initial conditions for the time derivative of ϕ_n^i are furthermore given by the initial interface shape:

$$\begin{array}{lcllcl} \phi_n^1(0) &=& 0 & \phi_n^2(0) &=& 0 \\ \\ \dot{\phi}_n^1(0) &=& -\dfrac{a(0) X_n \omega_n^2}{g k_n \sinh k_n h_1} & \dot{\phi}_n^2(0) &=& +\dfrac{a(0) X_n \omega_n^2}{g k_n \sinh k_n h_2} \end{array} \tag{12}$$

Since the shape of the interface is only a function of $\dot{\phi}_n^i$, one only needs to calculate $\dot{\phi}_n^i$ from the previous system of ODE:

$$\begin{array}{lcl} \dot{\phi}_n^1 &=& -\dfrac{X_n \omega_n^2}{g k_n \sinh k_n h_1}\left(a(t) - \displaystyle\int_0^t a(\tau)\omega_n \sin\omega_n(t-\tau)d\tau\right) \\ \\ \dot{\phi}_n^2 &=& +\dfrac{X_n \omega_n^2}{g k_n \sinh k_n h_2}\left(a(t) - \displaystyle\int_0^t a(\tau)\omega_n \sin\omega_n(t-\tau)d\tau\right) \end{array} \tag{13}$$

2.4 Interface shape for two longitudinal accelerations

2.4.1 Heaviside acceleration

In this case, the acceleration $a(t)$ is given by:

$$a(t) = \begin{cases} 0 & t < 0 \\ a_0 & t \geq 0 \end{cases} \tag{14}$$

This gives the following shape for the interface:

$$\xi = \frac{a_0}{g}\left(x - \frac{L}{2} + \sum_{n\geq 0} \frac{4}{Lk_{2n+1}^2}\ \cos(\omega_{2n+1}t)\ \cos(k_{2n+1}x)\right) \tag{15}$$

2.4.2 Square step

In this case, the acceleration $a(t)$ is given by:

$$a(t) = \begin{cases} 0 & t < 0 \\ a_0 & 0 \leq t \leq t_0 \\ 0 & t > t_0 \end{cases} \tag{16}$$

This gives the following shape for the interface:

$$\xi = \begin{cases} \dfrac{a_0}{g}\left(x - \dfrac{L}{2} + \displaystyle\sum_{n\geq 0} \dfrac{4}{Lk_{2n+1}^2}\ \cos(\omega_{2n+1}t)\ \cos(k_{2n+1}x)\right) & \text{when } t \leq t_0 \\ \dfrac{a_0}{g}\left(-\displaystyle\sum_{n\geq 0} \dfrac{8}{Lk_{2n+1}^2}\ \sin\left(\omega_{2n+1}\left(t - \dfrac{t_0}{2}\right)\right) \sin\left(\omega_{2n+1}\dfrac{t_0}{2}\right)\ \cos(k_{2n+1}x)\right) & \text{when } t \geq t_0 \end{cases} \tag{17}$$

3 Test-case description

We give in this section the different relevant parameters for the simulation. Here the length of the rectangular cavity is $L = 1$ m. The height of the box is $2L$. The heights of each layer are respectively $h_1 = h_2 = 1$ m. The density of the gas is $\rho_1 = 1$ kg/m^3, the density of the liquid is $\rho_2 = 1000$ kg/m^3. Boundary conditions on the for walls are slip boundary conditions. The initial condition is a hydrostatic pressure distribution in the whole field. Gravity is $g = -10$ m/s^2. The longitudinal acceleration amplitude is $a_0 = 0.1$ m/s^2. The initial interface is located at $y = 0$ (see figure 1).

The mesh used for the simulations must be uniform in Cartesian coordinates with 40×80 and 80×160 cells. Two test cases are proposed:

- case 1: Heaviside test case, the run must be done up to the dimensional time of 50 s for both meshes.
- case 2: Square test case with $t_0 = 2\pi/\omega_1$ only for the finer mesh

For both test cases, a comparison between the analytical and numerical locations of the interface at $x = 0$ and $x = L$ should be provided. We give respectively on figures 3

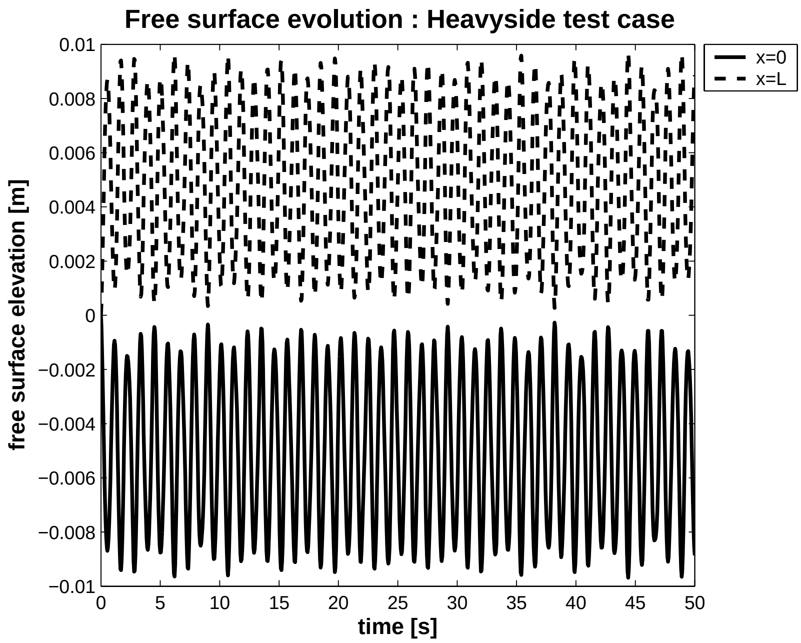

Figure 2: Analytical solution for the Heaviside acceleration test problem of section 2.4.1.

and 2, the analytical solutions corresponding to test-cases 1 and 2.

Figure 4 shows an example of the numerical results obtained by Chanteperdrix (2004) compared to the analytical solution of the first test-case. These results have been obtained on the finer mesh. The numerical diffusion due to long calculation time appears clearly. Consequently, it is recommended to calculate the following error, E, between numerical and analytical solution according to:

$$E(t) = \frac{1}{N_x} \sum_{i=1}^{N_x} \left| \xi_{\text{num}}(x_i, t) - \xi_{\text{th}}(x_i, t) \right| , \tag{18}$$

where N_x is the number of cells in the x-direction, ξ_{num} is the numerical surface elevation and ξ_{th} the theoretical one given by (15) or by (17).

References

Chanteperdrix, G. 2004. *Modélisation et Simulation Numérique des Ecoulements Diphasiques à Interfaces Libres. Application à l'étude des mouvements de liquides dans les réservoirs de véhicules spatiaux.* Ph.D. thesis, Ecole Nationale Supérieure de l'Aéronautique et de l'Espace.

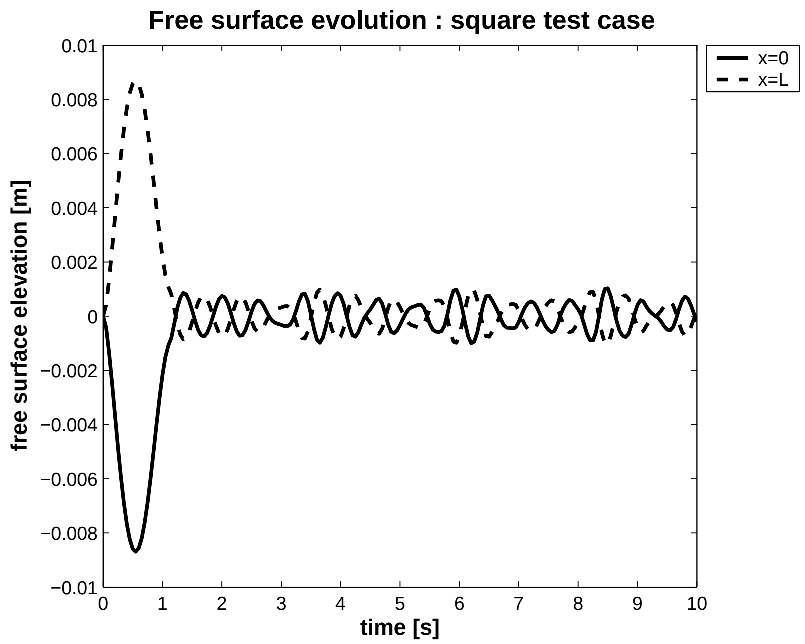

Figure 3: Analytical solution for the square step acceleration test problem of section 2.4.2.

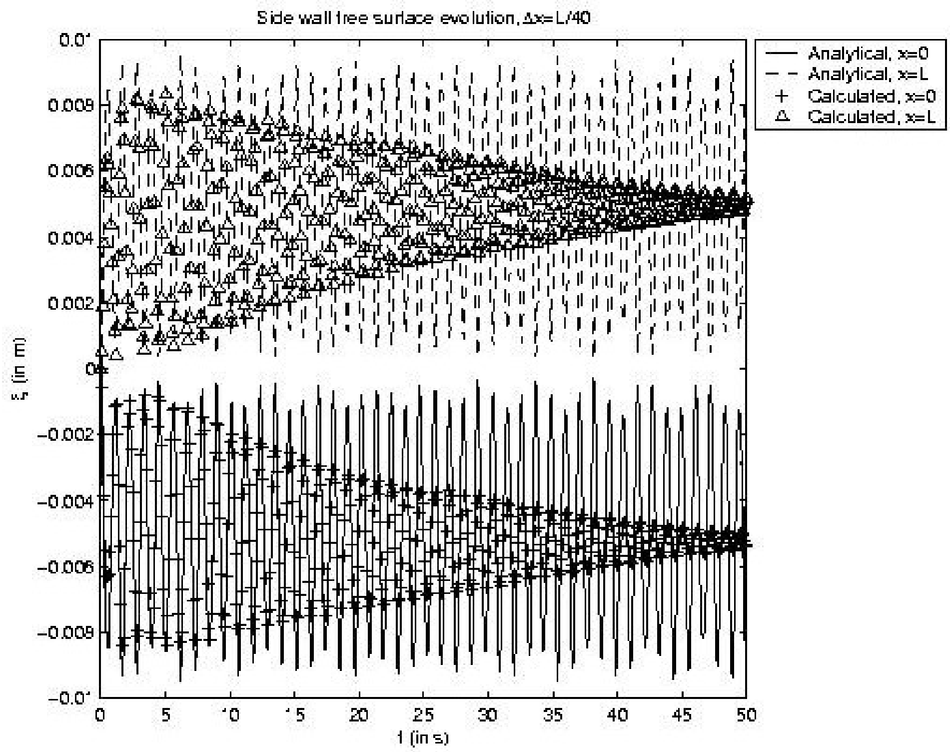

Figure 4: Comparison between the analytical and the numerical solution for the Heaviside acceleration test problem of section 2.4.1 (after Chanteperdrix, 2004).